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Contents

Part I

Integrals

1 Introduction

Integral calculus consists of

- **definite integral**, the calculation of a certain number, related to the area under a curve
- **indefinite integral**, the process of finding a function given its derivative

The definite and indefinite integrals are connected via the **(First) Fundamental Theorem of Calculus** (FToC.)

2 The definite integral

Let $f(x) : [a, b] \to \mathbb{R}$ be a bounded function. To calculate the area between the graph of the function and the x axis we divide the segment $[a, b]$ into n equal parts, with endpoints $x_0 = a, x_1, \dots, x_n = b$. Let $\Delta x = x_{i+1} - x_i$.

The area is divided into n strips, $S_1, S_2, \dots S_n$, with equal width $\Delta x = \dfrac{b-a}{n}$.

We choose a random point x_i^* in each strip and compute the value of f at those points.

We approximate the area of each strip by that of a rectangle whose base is Δx and whose height is $f(x_1^*)$, $f(x_2^*)$,..., $f(x_n^*)$. The area of each rectangle equals $f(x_i^*) \times \Delta x$.

Thus, the estimate of the total area is

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

This is a **Riemann integral sum**.

Definition 1. *Let $f(x) : [a, b] \to \mathbb{R}$ be a bounded function. Consider*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

where x_i^* and Δx are defined as above. If

1. *the limit exists and is finite*
2. *the limit does not depend on the choice of x_i^**

then f is called an ***integrable*** *function and the value of the limit is called the* ***definite integral of*** *$f(x)$* ***from*** *a* ***to*** *b and denoted by*

$$\int_a^b f(x)\,\mathrm{d}x$$

that is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The symbol $\int$ is the **integral sign**. $f(x)$ is the **integrand** and a and b are the **limits of integration**; a is the **lower limit** and b is the **upper limit**. $\mathrm{d}x$ represents the length $\Delta x = (b-a)/n$

as $n \to +\infty$. It also indicates that the independent variable is x. The calculation of an integral is called **integration**.

The definite integral is a number; it does not depend on x. In fact, we could use any letter in place of x and the value of the integral wouldn't change.

3 Areas & the definite integral

1. If $f \geq 0$, $\int_a^b f(x)\,\mathrm{d}x$ **equals the area** of the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$.

2. If $f \leq 0$, $\int_a^b f(x)\,\mathrm{d}x$ **equals the negative of the area** of the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$.

3. When f takes both positive and negative values, then the integral is equal to the areas above the x-axis *minus* the areas below the x-axis.

Thus, in general **the definite integral is not an area**.

4 Exercises

Exercise 1. Sketch the graph of

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.1 & 0 \leq x < 1 \\ 0.2 & 1 \leq x < 4 \\ 0.3 & 4 \leq x < 5 \\ 0 & x \geq 5 \end{cases}$$

and find

$$\int\limits_2^5 f(x)\,\mathrm{d}x \qquad \int\limits_0^5 f(x)\,\mathrm{d}x \qquad \int\limits_{-10}^{10} f(x)\,\mathrm{d}x$$

Exercise 2. Sketch a graph of the integrand and evaluate the following definite integral.

$$\int_0^4 f(x)\,\mathrm{d}x \quad \text{where} \quad f(x) = \begin{cases} 2 & x < 2 \\ 2x - 2 & x \geq 2 \end{cases}$$

5 Integrable functions

Obviously, since the definition of a definite integral starts from a bounded function, the following theorem holds:

Theorem 1 (Necessary condition for integrability). *If f is integrable on $[a, b]$ then it is bounded.*

Theorem 2 (Sufficient conditions for integrability). *On an interval $[a, b]$*

1. ***continuous*** *functions*
2. ***bounded discontinuous*** *functions with a* ***finite number of discontinuities***
3. ***monotonic*** *functions*

are integrable.

Remark. Boundedness alone is not sufficient for integrability.

Remark. Point 2 of theorem 2 implies that the value of f at a single point or at a finite number of points does not matter for the value of the definite integral.

6 Properties of the definite integral

1. integration interval with zero length: $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$
2. linearity with respect to the integration interval: $\displaystyle\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$.
3. swap of limits: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
4. linearity with respect to the integrand:
   (a) $\displaystyle\int_a^b \big[f(x) + g(x)\big]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$
   (b) $\displaystyle\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$, ($k$ is a constant)
5. monotonicity: if $f \le g$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$. In particular, since
$$-\big|f(x)\big| \le f(x) \le \big|f(x)\big|$$
we have
$$-\int_a^b \big|f(x)\big| \le \int_a^b f(x) \le \int_a^b \big|f(x)\big|$$

and finally

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b \left|f(x)\right|\mathrm{d}x$$

7 The mean value theorem

Definition 2. *If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ then the* ***mean value*** *of f in $[a, b]$ is*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Theorem 3. *If $f : [a, b] \to \mathbb{R}$ is continuous then there exists $c \in [a, b]$ such that*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = f(c)$$

that is, f takes its mean value at least once on $[a, b]$.

Proof. Since f is continuous is also integrable on $[a, b]$. In addition, by Weierstrass theorem continuity implies the existence of both $m = \min f$ and $M = \max f$ over $[a, b]$:

$$m \leq f(x) \leq M$$

By the monotonicity property, we have

$$\int_a^b m\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b M\,\mathrm{d}x.$$

Since the integral of a constant function over $[a, b]$ is just the constant times $b - a$ we have

$$(b-a)m \leq \int_a^b f(x)\,\mathrm{d}x \leq (b-a)M$$

which, divided by $b - a$ gets

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq M.$$

Finally, continuity of f on $[a, b]$ implies f takes on all values between m and M at least once in $[a, b]$ (intermediate value property) and therefore there must exist a number c such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

□

8 The fundamental theorem of calculus

Definition 3. *If $F : (a,b) \to \mathbb{R}$ is differentiable $\forall x \in (a,b)$ and*

$$F'(x) = f(x)$$

then F is an ***antiderivative*** *of f in (a,b).*

Antiderivatives are usually denoted by capital letters.

Theorem 4 (Fundamental Theorem of Calculus). *Let F be an antiderivative of f on $[a,b]$. If f is integrable on $[a,b]$ then*

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a)$$

$F(b) - F(a)$ is also written $F(x)\Big|_a^b$

9 Exercises

Exercise 3 (Fundamental Theorem of Calculus). Find the value of the following definite integrals using the FToC, or explain why it does not exist.

1. $\displaystyle\int_2^7 4\,\mathrm{d}x$
2. $\displaystyle\int_1^5 (2x)\,\mathrm{d}x$
3. $\displaystyle\int_1^5 (2x+3x^2)\,\mathrm{d}x$
4. $\displaystyle\int_1^5 (-2x-3x^2)\,\mathrm{d}x$
5. $\displaystyle\int_0^2 e^x\,\mathrm{d}x$
6. $\displaystyle\int_{-2}^2 x^3\,\mathrm{d}x$
7. $\displaystyle\int_2^4 \tfrac{1}{x^2}\,\mathrm{d}x$
8. $\displaystyle\int_{-2}^4 \tfrac{1}{x}\,\mathrm{d}x$
9. $\displaystyle\int_{-1}^2 f(x)\,\mathrm{d}x$ where $f(x)=\begin{cases}-x & x\le 0\\ x^2 & x>0\end{cases}$

Exercise 4 (Exam July 2012). Given the function

$$f(x)=\begin{cases}0 & x<0\\ 5 & 0\le x\le 5\\ 15 & x>5\end{cases}$$

find

$$\int_{-4}^{15} f(x)\ \mathrm{d}x$$

and

$$\int_{-4}^{15} \frac{f(x)}{x+5}\ \mathrm{d}x$$

Exercise 5. Prove that if f is continuous on $[a,b]$ and $\int_a^b f(x)\,\mathrm{d}x=0$, then there exists $c\in[a,b]$ such that $f(c)=0$.

Exercise 6. f is integrable and non-negative on $[a,b]$. Prove that if

$$\int_a^b f(x)\,\mathrm{d}x=0$$

then $f(x)=0$ at each point of continuity of f.

Exercise 7. Let $f(x)=\big|2-|x-3|\big|$. Find $\int_0^8 f(x)\,\mathrm{d}x$.

Exercise 8. Prove that

$$1\le\int_0^1 e^{x^2}\,\mathrm{d}x\le\frac{e+1}{2}$$

10 Indefinite integration

Theorem 5 (The family of antiderivatives). *If $F(x)$ is an antiderivative of $f(x)$ on (a,b), then **all the antiderivatives** of f on (a,b) have the form*

$$F(x) + c,$$

where c is an arbitrary constant.

Remark. The hypotheses about the interval cannot be removed. The theorem is false if we're allowed to consider any domain. For example, if we consider $f(x) = 1/x$ for $x \neq 0$, we have $F(x) = \ln(x)$ for $x > 0$ and $F(x) = \ln(-x)$ for $x < 0$, that is $F(x) = \ln(|x|)$. While every $F(x) + C$ (with any C) is an antiderivative of f, there are infinitely many other antiderivatives that are not of the form $F(x) + C$. For example,

$$\tilde{F}(x) = \begin{cases} \ln(x) + 3 & x > 0 \\ \ln(-x) + 2 & x < 0 \end{cases}$$

has f as its derivative and, obviously, $\tilde{F} \neq F(x) + C$ for any C. By changing 3 and 2 in the $\tilde{F}$ definition one can get infinitely many antiderivatives with the same property.

Definition 4. *The set of all antiderivatives of f on (a,b) is the **indefinite integral** of f and is denoted by*

$$\int f(x)\,dx$$

If F is any antiderivative of f on (a,b) then

$$\int f(x)\,dx = F(x) + c$$

11 Basic integration rules

Some basic formulas:

- $\displaystyle\int (ax+b)\,\mathrm{d}x = \frac{1}{2}ax^2 + bx + C$
- $\displaystyle\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C \qquad (a \neq -1)$
- $\displaystyle\int e^x\,\mathrm{d}x = e^x + C$
- $\displaystyle\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C \qquad (a > 0, a \neq 1)$
- $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln x + C \qquad x \in (0, +\infty)$
- $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \ln(-x) + C \qquad x \in (-\infty, 0)$
- $\displaystyle\int \cos(x)\,\mathrm{d}x = \sin(x) + C$
- $\displaystyle\int \sin(x)\,\mathrm{d}x = -\cos(x) + C$

12 Common patterns

- Linearity
$$\boxed{\int (\alpha f + \beta g)\,\mathrm{d}x = \alpha \int f\,\mathrm{d}x + \beta \int g\,\mathrm{d}x}$$
- Square
$$\boxed{\int f' \cdot f\,\mathrm{d}x = \frac{1}{2} \cdot f^2 + C}$$
- Power
$$\boxed{\int f' \cdot f^a\,\mathrm{d}x = \frac{1}{a+1} \cdot f^{a+1} + C}$$
- Logarithm
$$\boxed{\int \frac{f'}{f}\,\mathrm{d}x = \ln f + C} \qquad \text{or} \qquad \boxed{\int \frac{f'}{f}\,\mathrm{d}x = \ln(-f) + C}$$
choose the one that makes f positive in the integration interval.
- Exponential
$$\boxed{\int f' e^f\,\mathrm{d}x = e^f + C}$$

13 Exercises

Exercise 9 (Basic rules and common patterns). Find the indefinite integral using basic rules and common patterns and check the results by differentiation

1. $\int (x+7)\,\mathrm{d}x$
2. $\int (13-x)\,\mathrm{d}x$
3. $\int (x^5+1)\,\mathrm{d}x$
4. $\int (8x^3-9x^2+4)\,\mathrm{d}x$
5. $\int \frac{1}{x^5}\,\mathrm{d}x$
6. $\int \frac{1}{x^6}\,\mathrm{d}x$
7. $\int \frac{1}{\sqrt{x}}\,\mathrm{d}x$
8. $\int \frac{x+6}{\sqrt{x}}\,\mathrm{d}x$
9. $\int \frac{x^2+2x-3}{x^4}\,\mathrm{d}x$
10. $\int (x+1)(3x-2)\,\mathrm{d}x$
11. $\int (2t^2-1)^2\,\mathrm{d}t$
12. $\int y^2\sqrt{y}\,\mathrm{d}y$
13. $\int (1+3z)z^2\,\mathrm{d}z$
14. $\int \mathrm{d}x$
15. $\int 28\,\mathrm{d}x$
16. $\int 2e^x\,\mathrm{d}x$
17. $\int (3x^2-e^x)\,\mathrm{d}x$
18. $\int 2x(x^2-4)\,\mathrm{d}x$
19. $\int x^2\sqrt{x^3-4}\,\mathrm{d}x$
20. $\int \frac{3x^2}{x^3-4}\,\mathrm{d}x$
21. $\int (1+2t)e^{t+t^2}\,\mathrm{d}t$
22. $\int \frac{(\ln x)^3}{x}\,\mathrm{d}x$

Exercise 10. Find the antiderivatives of the following functions satisfying the condition on the right

1. $f(x) = 8x^3 - 9x^2 + 4$, $F(1) = 2$
2. $f(x) = \frac{1}{x^5}$, $F(1) = 1$
3. $f(x) = \frac{x+6}{\sqrt{x}}$, $F(4) = 24$
4. $f(x) = (1+3x)x^2$, $F(0) = 2$
5. $f(x) = 1$, $F(3) = 5$
6. $f(x) = 3x^2 - e^x$, $F(0) = 1$
7. $f(x) = 1/x$, $F(2) = 3$

Exercise 11 (Exam June 2012). A good G is sold at a certain price. The total cost of production of x units of G is $C(x)$, $x \geq 0$. $C(0) = 2000$ represents the fixed costs. The marginal production cost is $C'(x) = 10 + 2x$. Find the total production cost $C(x)$. Find the number of units for which fixed costs equal variable costs. If the marginal revenue for x units is $R'(x) = 8000(1-x)e^{-x}$, find the profit $P(x) = R(x) - C(x)$ assuming that $R(0) = 0$.

14 Integration by parts

Theorem 6 (Integration by parts). *If u and v are differentiable functions then*

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

Example 1. Consider

$$\int xe^x \, \mathrm{d}x$$

To apply integration by parts we have to choose who's u and who's $\mathrm{d}v$. Let's set

$$u = x \qquad \mathrm{d}v = e^x \, \mathrm{d}x$$

Then

$$\mathrm{d}u = \mathrm{d}x \qquad v = e^x$$

and the formula yields:

$$\int xe^x \, \mathrm{d}x = xe^x - \int e^x \cdot 1 \, \mathrm{d}x = xe^x - e^x + C$$

Example 2. Consider

$$\int x \ln x \, \mathrm{d}x$$

Let's set

$$\mathrm{d}v = x \, \mathrm{d}x \qquad u = \ln x$$

Then

$$v = \frac{x^2}{2} \qquad \mathrm{d}u = \frac{1}{x} \, \mathrm{d}x$$

and the formula yields:

$$\int x \ln x \, \mathrm{d}x = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, \mathrm{d}x = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, \mathrm{d}x = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

Example 3. To find

$$\int \ln x \, \mathrm{d}x$$

rewrite the integrand as $1 \cdot \ln x$:

$$\int 1 \cdot \ln x \, \mathrm{d}x$$

and set

$$\mathrm{d}v = \mathrm{d}x \qquad u = \ln x$$

Then

$$v = x \qquad \mathrm{d}u = \frac{1}{x} \, \mathrm{d}x$$

so that

$$\int \ln x \, \mathrm{d}x = x \ln x - \int x \cdot \frac{1}{x} \, \mathrm{d}x = x \ln x - \int \mathrm{d}x = x \ln x - x + C$$

15 The substitution rule

Theorem 7. *If $x = g(u)$ is a continuously differentiable function:*

$$\int f(x)\,\mathrm{d}x = \int f\big(g(u)\big)\,g'(u)\,\mathrm{d}u$$

Example 4. Find

$$\int x\sqrt{x-1}\,\mathrm{d}x$$

using the substitution $u = x - 1$. We have $x = u + 1$ and $\mathrm{d}x = \mathrm{d}u$. We can rewrite the integral as

$$\int x\sqrt{x-1}\,\mathrm{d}x = \int (u+1)\sqrt{u}\,\mathrm{d}u = \int u^{3/2} + u^{1/2}\,\mathrm{d}u = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

and going back to x

$$\int x\sqrt{x-1}\,\mathrm{d}x = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

Example 5. Find:

$$\int \frac{1-3x}{3x+2}\,\mathrm{d}x$$

using $u = 3x + 2$. We get $x = u/3 - 2/3$ and $\mathrm{d}x = \mathrm{d}u/3$. We can rewrite the integral as

$$\int \frac{1-3x}{3x+2}\,\mathrm{d}x = \int \frac{1 - 3\cdot\dfrac{u-2}{3}}{u}\frac{\mathrm{d}u}{3} = \frac{1}{3}\int \frac{3-u}{u}\,\mathrm{d}u = \frac{1}{3}\int \frac{3}{u} - 1\,\mathrm{d}u = \ln|u| - \frac{u}{3} + C$$

and going back to x

$$\int \frac{1-3x}{3x+2}\,\mathrm{d}x = \ln|3x+2| - \frac{3x+2}{3} + C$$

16 Substitution for definite integrals

Example 6. Consider:

$$\int_1^2 \frac{2x+1}{(2x+3)^2}\,\mathrm{d}x$$

Set $u = 2x + 3$. From this we get $\mathrm{d}u = 2\,\mathrm{d}x$ and $2x = u - 3$. We rewrite the integral as

$$\int \frac{2x+1}{(2x+3)^2}\,\mathrm{d}x = \int \frac{u-2}{u^2}\cdot\frac{\mathrm{d}u}{2} = \frac{1}{2}\int\left(\frac{1}{u} - \frac{2}{u^2}\right)\mathrm{d}u = \frac{1}{2}\left(\ln|u| + \frac{2}{u}\right) + C$$

Since $u(1) = 5$ and $u(2) = 7$ we have

$$\int_1^2 \frac{2x+1}{(2x+3)^2}\,\mathrm{d}x == \frac{1}{2}\left(\ln|u| + \frac{2}{u}\right)\Bigg|_5^7 == \frac{1}{2}\left(\ln 7 + \frac{2}{7} - \ln 5 - \frac{2}{5}\right)$$

Sometimes, after setting $u = g(x)$ one can immediately get $\mathrm{d}u = g'(x)\,\mathrm{d}x$ and substitute into the integral

17 Exercises

Exercise 12 (Integration by parts). Find the following indefinite integrals using integration by parts

1. $\int xe^{2x}\,\mathrm{d}x$
2. $\int x^5 \ln x\,\mathrm{d}x$
3. $\int xe^{-3x}\,\mathrm{d}x$
4. $\int \ln x\,\mathrm{d}x$ (set $u = \ln x$ and $\mathrm{d}v = \mathrm{d}x$)
5. $\int x^2 e^{-x}\,\mathrm{d}x$ (use integration by parts twice)
6. $\int x \sin x\,\mathrm{d}x$

Exercise 13 (Substitution rule). Find the following indefinite integrals using the given substitution

1. $\int x\sqrt{x-1}\,\mathrm{d}x,\ u = \sqrt{x-1}$
2. $\int 2x(x^2+1)^4\,\mathrm{d}x,\ u = x^2+1$
3. $\int \frac{(\ln x)^2}{x}\,\mathrm{d}x,\ u = \ln x$
4. $\int \frac{e^x}{e^x+2}\,\mathrm{d}x,\ u = e^x$

Exercise 14 (Substitution in definite integrals). Find the following definite integrals using the given substitution

1. $\int_1^4 \frac{1}{x+\sqrt{x}}\,\mathrm{d}x,\ u = \sqrt{x}$
2. $\int_1^e \frac{(\ln x)^2}{x}\,\mathrm{d}x,\ u = \ln x$
3. $\int_0^2 \frac{1}{(2x+3)^2}\,\mathrm{d}x,\ u = 2x+3$
4. $\int_0^4 \frac{x}{x^2+1}\,\mathrm{d}x,\ u = x^2+1$

Exercise 15 (Midterm 2012). Find $F(x)$, the antiderivative of $f(x) = xe^{-\frac{1}{2}x}$ such that $F(0) = -1$

Exercise 16 (*). Find $\int \frac{2e^x}{e^{2x}-1}\,\mathrm{d}x$

Exercise 17 (*,Midterm 2013). Consider the function $f(x) = \frac{x}{\sqrt{3x^2+1}}$. Using the substitution $y = 1+3x^2$ find all the antiderivatives of f. Find $\int_0^1 f(x)\,\mathrm{d}x$ using directly the variable y on the corresponding interval.

18 Improper integrals

Definition 5. *Let f be integrable over (a, b) for every $b > a$. If the limit*

$$L = \lim_{b \to +\infty} \int_a^b f(x)\ \mathrm{d}x$$

exists and is finite then we define

$$\int_a^{+\infty} f(x)\ \mathrm{d}x = L$$

*and we say the **improper integral converges**, otherwise it diverges.*

Definition 6. *Let f be integrable over (a, b) for every $a < b$. If the limit*

$$L = \lim_{a \to -\infty} \int_a^b f(x)\ \mathrm{d}x$$

exists and is finite then we define

$$\int_{-\infty}^{b} f(x)\ \mathrm{d}x = L$$

*and we say the **improper integral converges**, otherwise it diverges.*

Definition 7. *Let f be integrable over (a, b) for every a and b. Choose a point c and consider the two integrals:*

$$\int_{-\infty}^{c} f(x)\ \mathrm{d}x \qquad \text{and} \qquad \int_{c}^{+\infty} f(x)\ \mathrm{d}x$$

*If **both** improper integrals converge, then the improper integral*

$$\int_{-\infty}^{+\infty} f(x)\ \mathrm{d}x$$

converges and

$$\int_{-\infty}^{+\infty} f(x)\ \mathrm{d}x = \int_{-\infty}^{c} f(x)\ \mathrm{d}x + \int_{c}^{+\infty} f(x)\ \mathrm{d}x$$

19 The family $f(x) = 1/x^p$

Theorem 8 (The family $f(x) = 1/x^p$). *Consider the integral*

$$\int_1^{+\infty} \frac{1}{x^p}\ \mathrm{d}x.$$

- *If $p > 1$ the improper integral converges*

$$\int_1^{+\infty} \frac{1}{x^p}\,\mathrm{d}x = \frac{1}{p-1}$$

- *If $p \leq 1$ the improper integral diverges*

$$\int_1^{+\infty} \frac{1}{x^p}\,\mathrm{d}x = +\infty$$

20 Convergence conditions

Theorem 9 (Comparison test). *If*

1. *f and g are integrable over $[a, b]$ for any $b > a$*
2. *$0 \leq f(x) \leq g(x)$ on $[a, +\infty)$*

then

$$\int\limits_a^{+\infty} f(x)\,\mathrm{d}x \leq \int\limits_a^{+\infty} g(x)\,\mathrm{d}x$$

Therefore:

- *if $\int f$ is not integrable then $\int g$ is not integrable*
- *if $\int g$ is integrable then $\int f$ is integrable*

Theorem 10 (Asymptotic comparison test). *If for $x \geq a$*

- *$f(x) \geq 0$*
- *$g(x) \geq 0$*
- *f and g are integrable on $[a, k]$ for all $k > a$*
- *$f(x) \sim g(x)$ as $x \to +\infty$*

then

$$\int\limits_a^{+\infty} f(x)\,\mathrm{d}x \quad \textit{is integrable} \iff \int\limits_a^{+\infty} g(x)\,\mathrm{d}x \quad \textit{is integrable}$$

Theorem 11 (Absolute convergence). *If*

- *$f : [a, +\infty) \to \mathbb{R}$ is integrable on $[a, b]$ for any $b > a$*
- *$|f|$ is integrable on $[a, +\infty)$*

then f is also integrable on $[a, +\infty)$ and

$$\left| \int_a^{+\infty} f(x) \, \mathrm{d}x \right| \leq \int_a^{+\infty} \big| f(x) \big| \, \mathrm{d}x$$

.

21 Exercises

Exercise 18 (Improper integrals). Evaluate the following integrals.

1. $\displaystyle\int_3^{+\infty} x^{-2}\,\mathrm{d}x$
2. $\displaystyle\int_0^{+\infty} \frac{1}{(x+1)^3}\,\mathrm{d}x$
3. $\displaystyle\int_1^{+\infty} e^{-x}\,\mathrm{d}x$
4. $\displaystyle\int_2^{+\infty} \frac{2}{\sqrt{x}}\,\mathrm{d}x$
5. $\displaystyle\int_0^{+\infty} xe^{-x^2}\,\mathrm{d}x.$
6. $\displaystyle\int_{-\infty}^{0} e^{2x}\,\mathrm{d}x$
7. $\displaystyle\int_{-\infty}^{0} e^{3x}\,\mathrm{d}x$

Exercise 19 (Convergence conditions). Based on one of the convergence criteria, state if the following integrals converge or diverge

1. $\displaystyle\int_3^{+\infty} \frac{2x}{5x^3+x^2+1}\,\mathrm{d}x$
2. $\displaystyle\int_3^{+\infty} \frac{3x^3+\sqrt{x}}{2x^3+x^2+\sqrt{x}}\,\mathrm{d}x$
3. $\displaystyle\int_3^{+\infty} \frac{1}{x\sqrt{x+1}}\,\mathrm{d}x$
4. $\displaystyle\int_3^{+\infty} \frac{x^2}{x^3+1}\,\mathrm{d}x$
5. $\displaystyle\int_3^{+\infty} \frac{\ln x}{x^3}\,\mathrm{d}x$
6. $\displaystyle\int_1^{+\infty} x^3e^{-x}\,\mathrm{d}x$

Exercise 20 (*,Midterm 2012). Given

$$f(x) = xe^{-\frac{1}{2}x}$$

find

$$\int_0^{+\infty} f(x)\,\mathrm{d}x$$

22 Integral functions

Definition 8 (Functions defined by integrals). *If f is an integrable function in $[a,b]$ and if $c \in (a,b)$ then*

$$F(x) = \int_c^x f(t)\ \mathrm{d}t$$

is the ***integral function*** *of $f(x)$ with lower limit c.*

23 The second FToC

Theorem 12 (Second Fundamental Theorem of Calculus).

1. *If $f : [a,b] \to \mathbb{R}$ is integrable then*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is continuous.

2. *If f is also continuous then $F(x)$ is differentiable and $F'(x) = f(x)$.*

Proof. Assume f is integrable and $x_0 \in [a,b]$. Then, F exists and

$$F(x) - F(x_0) = \int\limits_a^x f(t)\,\mathrm{d}t - \int\limits_a^{x_0} f(t)\,\mathrm{d}t = \int\limits_{x_0}^x f(t)\,\mathrm{d}t$$

Being integrable, f is also bounded on $[a,b]$ and therefore there exists $M \in \mathbb{R}$ s.t. $\big|f(x)\big| < M$ for every $x \in [a,b]$. Thus

$$\big|F(x) - F(x_0)\big| = \left|\int\limits_{x_0}^x f(t)\,\mathrm{d}t\right| \leq \int\limits_{x_0}^x \big|f(t)\big|\,\mathrm{d}t \leq M|x - x_0|$$

which shows that if $x \to x_0$ then $F(x) \to F(x_0)$.

Suppose now f is also continuous. Then the difference quotient for F (in $[a,b]$) is

$$\frac{F(x+h) - F(x)}{h} = \frac{\int\limits_x^{x+h} f(t)\,\mathrm{d}t}{h}$$

By the mean value theorem there exists $c \in [x, x+h]$ such that

$$\int\limits_x^{x+h} f(t)\,\mathrm{d}t = h \cdot f(c)$$

which means

$$\frac{F(x+h) - F(x)}{h} = \frac{h \cdot f(c)}{h} = f(c).$$

Now, taking the limit $h \to 0$ we also have $c \to x$ (since c is bounded between x adn $x+h$). That is

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x).$$

which shows the derivative exists and it's equal to $f(x)$. $\square$

Remark. Things we know about F (if f is continuous and differentiable):

- $F(c) = 0$
- $F'(x) = f(x)$; if $f > 0$, F is increasing; if $f < 0$, F is decreasing
- $F''(x) = f'(x)$; if $f' > 0$, F is convex; if $f' < 0$, F is concave
- $\lim\limits_{x\to+\infty} F(x) = \int\limits_c^{+\infty} f(t)\,\mathrm{d}t$; thus F has limit if the improper integral exists

Example 7. If $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \end{cases}$, find $F(x) = \int_0^x f(t)\,\mathrm{d}t$ in the interval $[0,2]$ and sketch a graph of F.

The graph of $F'(x)$ is that of $f(x)$ (see Figure 1, left). $F(x)$ is an increasing function because its derivative is positive and looking at the lower limit of the integral we see that $F(0) = 0$.

On $[0,1]$, we have

$$F(x) = \int\limits_0^x f(t)\,\mathrm{d}t = \int\limits_0^x t\,\mathrm{d}t = \left.\frac{t^2}{2}\right|_0^x = \frac{x^2}{2}$$

If $1 \le x \le 2$ then

$$F(x) = \int\limits_0^x f(t)\,\mathrm{d}t = \int\limits_0^1 f(t)\,\mathrm{d}t + \int\limits_1^x f(t)\,\mathrm{d}t = \left.\frac{t^2}{2}\right|_0^1 + \int\limits_1^x 1\,\mathrm{d}t = \frac{1}{2} + t\Big|_1^x = \frac{1}{2} + x - 1 = x - \frac{1}{2}$$

Therefore,

$$F(x) = \begin{cases} \dfrac{x^2}{2} & 0 \le t \le 1 \\ x - \dfrac{1}{2} & 1 < x \le 2 \end{cases}$$

The graph of $F(x)$ is shown on the right in figure 1.

Example 8. Draw a graph of

$$F(x) = \int_{-\infty}^x f(t)\ \mathrm{d}t \qquad \text{where} \qquad f(x) = \begin{cases} 0 & x < 0 \\ 2 & 0 \le x \le 3 \\ 4 & x > 3 \end{cases}$$

The graph of f is shown in figure 2. For $x \le 0$ we have

$$F(x) = \int\limits_{-\infty}^x f(t)\,\mathrm{d}t = \int\limits_{-\infty}^x 0\,\mathrm{d}t = 0$$

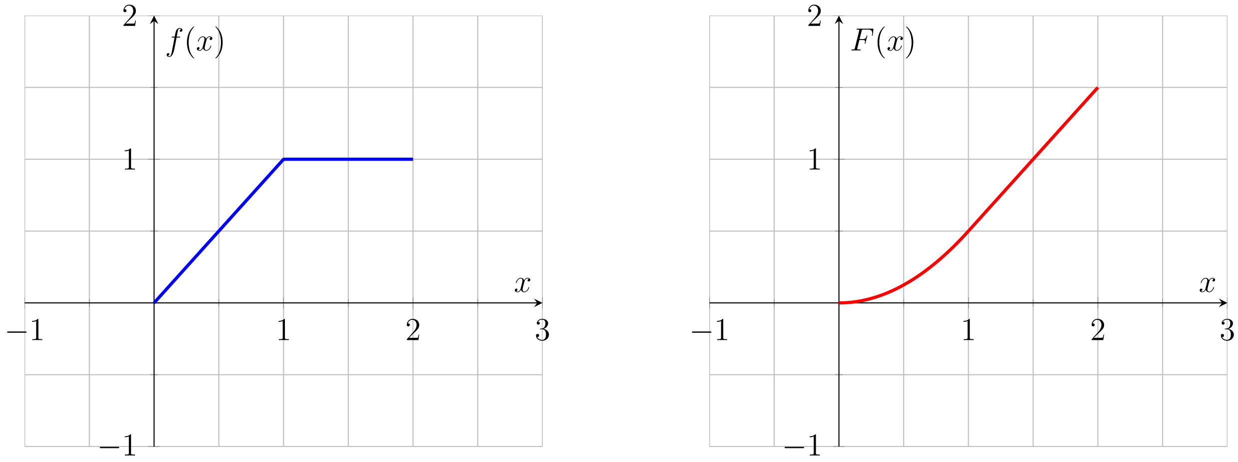


Figure 1: $f(x)$ (on the left) and its integral function $F(x) = \int_0^x f(t)\, \mathrm{d}t$ on the right.

If $0 \leq x \leq 3$ then

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{0} f(t)\,\mathrm{d}t + \int_{0}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{0} 0\,\mathrm{d}t + \int_{0}^{x} 2\,\mathrm{d}t = 0 + 2t\Big|_0^x = 2x$$

Finally, if $x > 3$ then

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{0} f(t)\,\mathrm{d}t + \int_{0}^{3} f(t)\,\mathrm{d}t + \int_{3}^{x} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{0} 0\,\mathrm{d}t + \int_{0}^{3} 2\,\mathrm{d}t + \int_{3}^{x} 4\,\mathrm{d}t \\
&= 0 + 6 + 4t\Big|_3^x \\
&= 6 + 4x - 12 = 4x - 6
\end{aligned}$$

The graph of F is shown in figure 2.

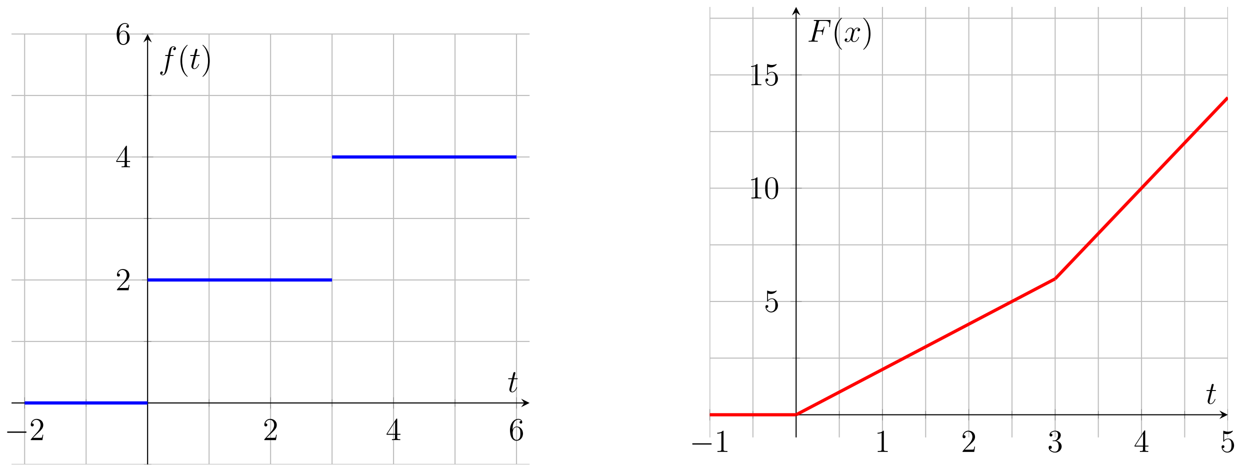


Figure 2: $f(t)$ (on the left) and its integral function $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ on the right.

24 Exercises

Exercise 21. Find the analytical expression for each of the following integral functions:

1. $F(x) = \int_0^x f(t)\,\mathrm{d}t$ where $f(x) = \begin{cases} x & 0 \le x \le 1 \\ x-1 & 1 < x \le 2 \end{cases}$

2. $F(x) = \int_0^x f(t)\,\mathrm{d}t$ where $f(x) = \begin{cases} 0 & x < 0 \\ 5 & 0 \le x \le 5 \\ 15 & x > 5 \end{cases}$

3. $F(x) = \int_0^x f(t)\,\mathrm{d}t$ where $f(x) = \begin{cases} e^x & x \le 0 \\ e^{-x} & x > 0 \end{cases}$

4. $F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$ where $f(x) = \begin{cases} e^x & x \le 0 \\ e^{-x} & x > 0 \end{cases}$

5. $F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$ where $f(x) = \begin{cases} 1/x^2 & x \le -1 \\ 1 & -1 < x < 1 \\ 1/x^2 & x \ge 1 \end{cases}$

Exercise 22 (MC). If the mean value of f on $[1,2]$ is 2 and the mean value of f on $[2,4]$ is 3 then the mean value of f on $[1,4]$ is

a) 5

b) 8/3

c) 8

d) none of the others

Exercise 23 (MC). If f is integrable on $[0,5]$ and $\int_1^3 f\,\mathrm{d}x = 3$ and $\int_1^5 f\,\mathrm{d}x = 3$ then $\int_3^5 f\,\mathrm{d}x$ is

a) 0

b) 3

c) 3/2

d) none of the others

Exercise 24. Prove that if f is continuous on $[a,b]$ and $\int_a^b f(x)\,\mathrm{d}x = 0$, then there exists $c \in [a,b]$ such that $f(c) = 0$. Is continuity a necessary hypothesis for the statement to be true?

Exercise 25. f is integrable and non-negative on $[a,b]$. Prove that if

$$\int_a^b f(x)\,\mathrm{d}x = 0$$

then $f(x) = 0$ at each point of continuity of f.

Exercise 26. Let $f(x) = \big|2 - |x-3|\big|$. Find $\int_0^8 f(x)\,\mathrm{d}x$.

Exercise 27. Prove that

$$1 \leq \int_0^1 e^{x^2}\,\mathrm{d}x \leq \frac{e+1}{2}$$

Exercise 28. Given

$$F(x) = \int_3^x \frac{4}{t^2-4}\,\mathrm{d}t$$

write the equation of the tangent to F at $x_0 = 3$.

Exercise 29. Let

$$F(x) = \int_1^x \frac{t+1}{t^2+1}\,\mathrm{d}t$$

Sketch a graph of F.

Exercise 30. Let

$$F(x) = \int_0^x t^4 e^{-t^2}\,\mathrm{d}t$$

Sketch a graph of F.

Exercise 31. Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying

$$\int_0^x f(t)\,\mathrm{d}t = \int_x^1 f(t)\,\mathrm{d}t, \qquad \forall x \in [0,1]$$

Prove that $f(x) = 0$, $\forall x \in [0,1]$.

Exercise 32 (MC). If

$$F(x) = \int_2^x \frac{\ln t}{e^t+1}\,\mathrm{d}t$$

and

$$G(x) = \int_3^x \frac{\ln t}{e^t+1}\,\mathrm{d}t$$

then

a) $F(x) = G(x) + k$, with $k \in \mathbb{R}$, $k > 0$

b) $G(x) = F(x)$

c) $G'(x) = F'(x) + k$, with $k \in \mathbb{R}$, $k \neq 0$

d) none of the others

Exercise 33 (MC). $f : \mathbb{R} \to \mathbb{R}$ is a continuous negative function and

$$F(x) = \int_2^x f(t)\,\mathrm{d}t$$

Then

a) $F(x) < 0$ for all $x \in \mathbb{R}$

b) $F(x) > 0$ for all $x > 2$

c) $F(x)$ is decreasing for all $x \in \mathbb{R}$

d) none of the others

Exercise 34 (TF). $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x+1 & x \geq 0 \\ x & x < 0 \end{cases}$$

a) $f(x)$ is integrable on $[-1, 1]$

b) $\int_0^2 f(x)\,\mathrm{d}x = 4$

c) $\int_{-1}^1 f(x)\,\mathrm{d}x$ does not exist

d) $F(x) = \int_0^x f(t)\,\mathrm{d}t$ is not defined for $x < 0$

e) $\int_{-c}^c f(x)\,\mathrm{d}x \neq 0$ for every $c > 0$

Exercise 35 (TF). $F : [a, b] \to \mathbb{R}$ is defined by

$$F(x) = \int\limits_c^x f(t)\,\mathrm{d}t$$

where $c \in (a, b)$ and $f : [a, b] \to \mathbb{R}$ is an integrable function

a) $F(x)$ is continuous at every $x \in (a, b)$

b) $F'(x) = f(x)$ at every $x \in (a, b)$

c) if the mean value of f in $[a, b]$ is positive then $F(x) > 0$ for every $x \in (a, b)$

d) if $f(x) < 0$ for some $x \in (a, b)$ then $F(x) < 0$ for some $x \in (a, b)$

Exercise 36 (TF). $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[a, b]$.

a) if the mean value of f is 0 then there exists $c \in (a, b)$ such that $f(c) = 0$

b) if the mean value of f is positive then there exists $c \in (a, b)$ such that $f(c) > 0$

c) if there exists $c \in (a, b)$ such that $f(c) > 0$ then the mean value of f is positive

d) if there exists $c \in (a, b)$ such that $f(c) = 0$ then the mean value of f is 0

e) if there exists $c \in (a, b)$ such that $f(c)$ equals the mean value of f, then f is continuous on $[a, b]$

f) if f is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that $f(c) =$ equals the mean value of f

Exercise 37 (MC). If

$$F(x) = \int_3^x \frac{t-4}{e^t+1}\,\mathrm{d}t$$

then

a) $F(x)$ is increasing at $x = 3$

b) 4 is a minimizer for F

c) $F(4) = 0$

d) none of the others

Exercise 38 (MC). If

$$F(x) = \int_2^{x^2} \frac{t}{e^t + 1}\,\mathrm{d}t$$

then $F'(x) =$

a) $2x\frac{x}{e^x+1}$

b) $\frac{x}{e^x+1}$

c) $2x\frac{x^2}{e^{x^2}+1}$

d) none of the others

Exercise 39 (MC). If

$$F(x) = \int_2^{x} \frac{\ln t}{e^t + 1}\,\mathrm{d}t$$

and

$$G(x) = \int_x^{3} \frac{\ln t}{e^t + 1}\,\mathrm{d}t$$

then

a) $G'(x) + F'(x) = 0$

b) $G(x) = F(x) + 1$

c) $G'(x) = F'(x)$

d) none of the others

Part II

Solutions

25 Solutions

1 The graph of the function is shown in figure 3. Since $\int\limits_2^5 f(x)\,\mathrm{d}x = \int\limits_2^4 f(x)\,\mathrm{d}x + \int\limits_4^5 f(x)\,\mathrm{d}x$ the total area is the sum two rectangles, one with base 2 and height 0.2 and the other with base 1 and height 0.3: $\int\limits_2^5 f(x)\,\mathrm{d}x = 0.2 \times (4-2) + 0.3 \times (5-4) = 0.7$. In the same way, we get

$$\int\limits_0^5 f(x)\,\mathrm{d}x = 0.1 \times 1 + 0.2 \times 3 + 0.3 \times 1 = 0.1 + 0.6 + 0.3 = 1. \tag{1}$$

Finally, since f is 0 anywhere except for $0 \leq x \leq 5$ then $\int\limits_{-10}^0 f(x)\,\mathrm{d}x = 0$ and $\int\limits_5^{10} f(x)\,\mathrm{d}x = 0$ which implies

$$\int\limits_{-10}^{10} f(x)\,\mathrm{d}x = \int\limits_{-10}^{0} f(x)\,\mathrm{d}x + \int\limits_{0}^{5} f(x)\,\mathrm{d}x + \int\limits_{5}^{10} f(x)\,\mathrm{d}x = 0 + \int\limits_{0}^{5} f(x)\,\mathrm{d}x + 0 = 1 \tag{2}$$

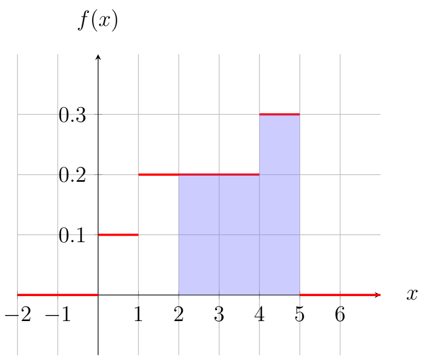


Figure 3: $f(x)$ of exercise 1 with the shaded area of the first integral

2 From the graph of $f(x)$ (figure 4) we see that $\int_0^4 f(x)\ \mathrm{d}x = 12$.

3

1. $$\int_2^7 4\,\mathrm{d}x = 4x\Big|_2^7 = 28 - 8 = 20$$

2. $$\int_1^5 2x\,\mathrm{d}x = x^2\Big|_1^5 = 5^2 - 1^2 = 25 - 1 = 24$$

3. $$\int_1^5 (2x+3x^2)\,\mathrm{d}x = \int_1^5 2x\,\mathrm{d}x + \int_1^5 x^2\,\mathrm{d}x = x^2\Big|_1^5 + x^3\Big|_1^5 = 5^2 - 1^2 + 5^3 - 1^3 = 25 - 1 + 125 - 1 = 148$$

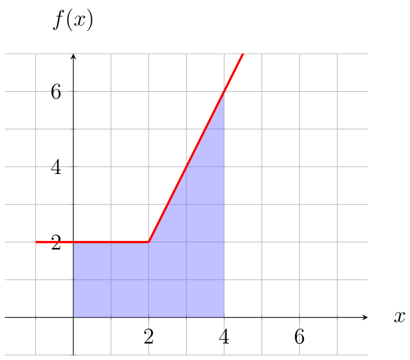


Figure 4: $f(x)$ of exercise 2

4.
$$\int_1^5 (-2x - 3x^2)\,\mathrm{d}x = -\int_1^5 (2x + 3x^2)\,\mathrm{d}x = -148$$
(see previous item.)

5.
$$\int_0^2 e^x\,\mathrm{d}x = e^x\Big|_0^2 = e^2 - e^0 = e^2 - 1$$

6.
$$\int_{-2}^2 x^3\,\mathrm{d}x = \frac{x^4}{4}\Big|_{-2}^2 = \frac{16}{4} - \frac{16}{4} = 0$$

7.
$$\int_2^4 \frac{1}{x^2}\,\mathrm{d}x = -\frac{1}{x}\Big|_2^4 = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

8. The function $1/x$ is not bounded in the interval $[-2, 4]$ since
$$\lim_{x \to 0^\pm} \frac{1}{x} = \pm\infty.$$
Therefore the integral does not exist.

9.
$$\int_{-1}^0 f(x)\,\mathrm{d}x = \int_{-1}^0 -x\,\mathrm{d}x + \int_0^2 x^2\,\mathrm{d}x = -\frac{x^2}{2}\Big|_{-1}^0 + \frac{x^3}{3}\Big|_0^2$$
$$= 0 - \left(-\frac{1}{2}\right) + \frac{8}{3} - 0 = \frac{19}{6}$$

4 Being discontinuous only at $x = 0$ and at $x = 5$, f is integrable. From
$$\int_{-4}^{15} f(x)\ \mathrm{d}x = \int_{-4}^{0} f(x)\ \mathrm{d}x + \int_0^5 f(x)\ \mathrm{d}x + \int_5^{15} f(x)\ \mathrm{d}x$$
we get
$$\int_{-4}^{15} f(x)\ \mathrm{d}x = \int_{-4}^{0} 0\ \mathrm{d}x + \int_0^5 5\ \mathrm{d}x + \int_5^{15} 15\ \mathrm{d}x = 0 + 25 + 150 = 175$$

For the second integral we have:

$$\begin{aligned}\int_{-4}^{15} \frac{f(x)}{x+5}\,\mathrm{d}x &= \int_{-4}^{0} 0\,\mathrm{d}x + \int_{0}^{5} \frac{5}{x+5}\,\mathrm{d}x + \int_{5}^{15} \frac{15}{x+5}\,\mathrm{d}x \\ &= 0 + 5\left|\ln(x+5)\right|_0^5 + 15\left|\ln(x+5)\right|_5^{15} \\ &= 0 + 5(\ln 10 - \ln 5) + 15(\ln 20 - \ln 10) \\ &= 0 + 5\ln 2 + 15\ln 2 = 20\ln 2\end{aligned}$$

5 Since f is continuous we can apply the mean value theorem. The mean value of f is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{1}{b-a}\cdot 0 = 0$$

and by the theorem there is at least one number c such that $f(c)$ equals the mean value, that is $f(c) = 0$.

6 Say c is a continuity point for f and suppose $f(c) > 0$. Then, by continuity, for every $\varepsilon > 0$ there is a δ such that if $0 < |x-c| < \delta$ then $\left|f(x) - f(c)\right| < \varepsilon$. Take $\varepsilon = f(c)/2$ and let δ_0 the corresponding δ. Then

$$\left|f(x) - f(c)\right| < \frac{f(c)}{2} \qquad |x-c| < \delta_0$$

that is

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2} \qquad |x-c| < \delta_0$$

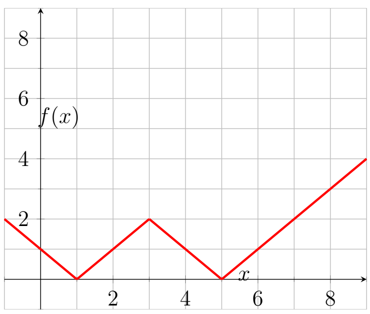
Thus, since f is non-negative, we have

$$\int_a^b f(x)\,\mathrm{d}x \geq \int_{c-\delta_0}^{c+\delta_0} f(x)\,\mathrm{d}x > \frac{f(c)}{2}\int_{c-\delta_0}^{c+\delta_0} \mathrm{d}x = f(c)\delta_0 > 0$$

which contradicts the hypothesis about the integral of f. Then, it must be that $f(x) = 0$ at every point of continuity.

Note that the function need not be 0 at isolated points. For example, the previous function can be equal to 1 at a point in $[a,b]$, say $(a+b)/2$. The integral would be 0 anyway, since the value of f at any single point does not influence that of the integral.

7 $f(x) \geq 0$ for every x so the integral equals the area under the curve. The graph of f is



and therefore

$$\int_0^8 f(x)\,\mathrm{d}x = \frac{1}{2} + 4 + \frac{9}{2} = 9$$

8 $f(x) = e^{x^2}$ is strictly increasing on $[0,1]$ and $f(0) = 1$ and $f(1) = e$. Thus we immediately have

$$1 \leq \int_0^1 e^{x^2}\,\mathrm{d}x.$$

To prove the second inequality, note that $f'(x) = 2xe^{x^2}$ and

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = 2e^{x^2}(1 + 2x2) > 0$$

therefore, f is convex on $[0,1]$ which means it's bounded by the chord passing through $(0,1)$ and $(1,e)$. Then the area under f is bounded by that of the trapezoid with sides 1, e and height 1, that is $(1+e)/2$.

9

1. $\displaystyle\int (x+7)\,\mathrm{d}x = \frac{x^2}{2} + 7x + C$
2. $\displaystyle\int (13-x)\,\mathrm{d}x = 13x - \frac{x^2}{2} + C$
3. $\displaystyle\int (x^5+1)\,\mathrm{d}x = \frac{x^6}{6} + x + C$
4. $\displaystyle\int (8x^3 - 9x^2 + 4)\,\mathrm{d}x = 2x^4 - 3x^3 + 4x + C$
5. $\displaystyle\int \frac{1}{x^5}\,\mathrm{d}x = \int x^{-5}\,\mathrm{d}x = -\frac{1}{4x^4} + C$
6. $\displaystyle\int \frac{1}{x^6}\,\mathrm{d}x = \int x^{-6}\,\mathrm{d}x = -\frac{1}{5x^5} + C$
7. $\displaystyle\int \frac{1}{\sqrt{x}}\,\mathrm{d}x = \int x^{-1/2}\,\mathrm{d}x = \frac{x^{1/2}}{\frac{1}{2}} + C = 2\sqrt{x} + C$
8. $\displaystyle\int \frac{x+6}{\sqrt{x}}\,\mathrm{d}x = \frac{2}{3}x^{3/2} + 12\sqrt{x} + C$ (use linearity)
9. $\displaystyle\int \frac{x^2+2x-3}{x^4}\,\mathrm{d}x = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C$ (use linearity)
10. $\displaystyle\int (x+1)(3x-2)\,\mathrm{d}x = \int (3x^2 + x - 2)\,\mathrm{d}x = x^3 + \frac{x^2}{2} - 2x + C$ (multiply)
11. $\displaystyle\int (2t^2-1)^2\,\mathrm{d}t = \int (4t^4 - 4t^2 + 1)\,\mathrm{d}t = \frac{4t^5}{5} - \frac{4t^3}{3} + t + C$ (expand the square)
12. $\displaystyle\int y^2\sqrt{y}\,\mathrm{d}y = \int y^{2+1/2}\,\mathrm{d}y = \int y^{5/2}\,\mathrm{d}y = \frac{y^{7/2}}{7/2} + C$ (rewrite as a single power)
13. $\displaystyle\int (1+3z)z^2\,\mathrm{d}z = \int z^2 + 3z^3\,\mathrm{d}z = \frac{z^3}{3} + \frac{3z^4}{4} + C$ (multiply)
14. $\displaystyle\int \mathrm{d}x = x + C$
15. $\displaystyle\int 28\,\mathrm{d}x = 28x + C$

16. $\int 2e^x \, \mathrm{d}x = 2e^x + C$

17. $\int (3x^2 - e^x) \, \mathrm{d}x = x^3 - e^x + C$ (use linearity)

18. $\int 2x(x^2-4) \, \mathrm{d}x = \frac{1}{2}(x^2-4)^2 + C$ (use pattern $f'f$ with $f = x^2 - 4$)

19. $\int x^2\sqrt{x^3-4} \, \mathrm{d}x = \frac{1}{3}\int 3x^2\sqrt{x^3-4} \, \mathrm{d}x = \frac{1}{3} \cdot \frac{(x^3-4)^{3/2}}{\frac{3}{2}} = \frac{2}{9}(x^3-4)^{3/2}$ (use pattern $f'f^a$ with $f = x^3 - 4$)

20. $\int \frac{3x^2}{x^3-4} \, \mathrm{d}x = \ln\left|x^3-4\right| + C$ (use pattern f'/f with $f = x^3 - 4$)

21. $\int (1+2t)e^{t+t^2} \, \mathrm{d}t = e^{t+t^2} + C$ (use pattern $f'e^f$ with $f = t + t^2$)

22. $\int \frac{(\ln x)^3}{x} \, \mathrm{d}x = \frac{1}{4}(\ln x)^4 + C$ (use pattern $f'f^a$ with $f = \ln x$)

10

1. Since $\int (8x^3 - 9x^2 + 4) \, \mathrm{d}x = 2x^4 - 3x^3 + 4x + C$, then $F(x) = 2x^4 - 3x^3 + 4x + C$ and $F(1) = 2-3+4+C = 3+C$. Since $F(1) = 2$, then $3+C = 2$, $C = -1$, $F(x) = 2x^4 - 3x^3 + 4x - 1$.

2. Since $\int \frac{1}{x^5} \, \mathrm{d}x = \int x^{-5} \, \mathrm{d}x = -\frac{1}{4x^4} + C$, then $F(x) = -\frac{1}{4x^4} + C$ and $F(1) = -\frac{1}{4} + C$. Since $F(1) = 1$, then $-\frac{1}{4} + C = 1$, $C = \frac{5}{4}$, $F(x) = -\frac{1}{4x^4} + \frac{5}{4}$

3. Since $\int \frac{x+6}{\sqrt{x}} \, \mathrm{d}x = \frac{2}{3}x^{3/2} + 12\sqrt{x} + C$, then $F(x) = \frac{2}{3}x^{3/2} + 12\sqrt{x} + C$ and $F(4) = \frac{16}{3} + 24 + C$. Since $F(4) = 24$ then $C = -\frac{16}{3}$, $F(x) = \frac{2}{3}x^{3/2} + 12\sqrt{x} - \frac{16}{3}$.

4. Since $\int (1+3x)x^2 \, \mathrm{d}x = \int x^2 + 3x^3 \, \mathrm{d}x = \frac{x^3}{3} + \frac{3x^4}{4} + C$, then $F(x) = \frac{x^3}{3} + \frac{3x^4}{4} + C$ and $F(0) = C$. Since $F(0) = 2$ then $C = 2$, $F(x) = \frac{x^3}{3} + \frac{3x^4}{4} + 2$.

5. Since $\int \mathrm{d}x = x + C$, then $F(x) = x + C$ and $F(3) = 3 + C$. Since $F(3) = 5$ then $3 + C = 5$, $C = 2$, $F(x) = x + 2$.

6. Since $\int (3x^2 - e^x) \, \mathrm{d}x = x^3 - e^x + C$ then $F(x) = x^3 - e^x + C$ and $F(0) = -1 + C$. Since $F(0) = 1$ then $-1 + C = 1$, $C = 2$, $F(x) = x^3 - e^x + 2$.

7. Since $\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + C$. To choose the input of ln note that F is defined for $x = 2$, i.e. in $(0, +\infty)$. Thus we have $\int \frac{1}{x} \, \mathrm{d}x = \ln x + C$. From

$$F(x) = \ln x + C$$
$$F(2) = 3$$

we get $F(2) = \ln 2 + C = 3$, which implies $C = 3 - \ln 2$. Thus

$$F(x) = \ln x + 3 - \ln 2 = \ln \frac{x}{2} + 3.$$

11 We know the derivative of $C(x)$ (the marginal cost) and its value at $x = 0$. $C(x)$ is the antiderivative of C' such that $C(0) = 2000$. Since

$$\int C'(x)\ \mathrm{d}x = \int (10 + 2x)\ \mathrm{d}x = 10x + x^2 + c$$

the antiderivatives of C' are $C(x) = 10x + x^2 + c$. Since $C(0) = 2000$, we have $C(x) = 10x + x^2 + 2000$. If fixed costs equal variable costs then $10x + x^2 = 2000$, and $x = 40$. To find R we start from $R'(x) = 8000(1 - x)e^{-x}$ and suppose $R(0) = 0$. The indefinite integral of R is

$$R(x) = \int 8000(1 - x)e^{-x}\ \mathrm{d}x$$

Using integration by parts we set $u = 1 - x$ and $\mathrm{d}v = e^{-x}\,\mathrm{d}x$. Thus, $\mathrm{d}u = -\,\mathrm{d}x$, $v = -e^{-x}$ and

$$R(x) = \int 8000(1 - x)e^{-x}\ \mathrm{d}x = 8000\left(-(1 - x)e^{-x} - \int e^{-x}\,\mathrm{d}x\right) = 8000xe^{-x} + K$$

Since $R(0) = 0$ then $K = 0$. The profit function is $P(x) = 8000xe^{-x} - 10x - x^2 - 2000$.

12 1. $\int xe^{2x}\,\mathrm{d}x$. Set $u = x$ and $\mathrm{d}v = e^{2x}\,\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = e^{2x}/2$. Thus

$$\int xe^{2x}\,\mathrm{d}x = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,\mathrm{d}x = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

2. $\int x^5 \ln x\,\mathrm{d}x$. Set $u = \ln x$ and $\mathrm{d}v = x^5\,\mathrm{d}x$. Then $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and $v = x^6/6$. Thus

$$\int x^5 \ln x\,\mathrm{d}x = \frac{x^6}{6}\ln x - \int \frac{x^6}{6}\frac{1}{x}\,\mathrm{d}x = \frac{x^6}{6}\ln x - \int \frac{x^5}{6}\,\mathrm{d}x = \frac{x^6}{6}\ln x - \frac{x^6}{36} + C$$

3. $\int xe^{-3x}\,\mathrm{d}x$. Set $u = x$ and $\mathrm{d}v = e^{-3x}\,\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = -e^{-3x}/3$. Thus

$$\int xe^{-3x}\,\mathrm{d}x = -x\frac{e^{-3x}}{3} - \int -\frac{e^{-3x}}{3}\,\mathrm{d}x = -x\frac{e^{-3x}}{3} - \frac{e^{-3x}}{9} + C$$

4. $\int \ln x\,\mathrm{d}x$. Set $u = \ln x$ and $\mathrm{d}v = \mathrm{d}x$. Then $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and $v = x$. Thus

$$\int \ln x\,\mathrm{d}x = x\ln x - \int x\frac{1}{x}\,\mathrm{d}x = x\ln x - \int \mathrm{d}x = x\ln x - x + C$$

5. $\int x^2e^{-x}\,\mathrm{d}x$. Set $u = x^2$ and $\mathrm{d}v = e^{-x}\,\mathrm{d}x$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$ and $v = -e^{-x}$. Thus

$$\int x^2e^{-x}\,\mathrm{d}x = -x^2e^{-x} - \int -e^{-x}2x\,\mathrm{d}x = -x^2e^{-x} + 2\int xe^{-x}\,\mathrm{d}x$$

Now, apply integration by parts again to solve the remaining integral. Set $u = x$ and $\mathrm{d}v = e^{-x}\,\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = -e^{-x}$. Thus

$$\begin{aligned}\int x^2e^{-x}\,\mathrm{d}x &= -x^2e^{-x} + 2\int xe^{-x}\,\mathrm{d}x \\ &= -x^2e^{-x} + 2\left[-xe^{-x} - \int -e^{-x}\,\mathrm{d}x\right] \\ &= -x^2e^{-x} + 2\left[-xe^{-x} - e^{-x}\right] + C \\ &= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C\end{aligned}$$

6. $\displaystyle\int x\sin x\,\mathrm{d}x$. Set $u = x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos x$. Thus

$$\int x\sin x\,\mathrm{d}x = -x\cos x - \int -\cos x\,\mathrm{d}x = -x\cos x + \sin x + C$$

13 1. $\displaystyle\int x\sqrt{x-1}\,\mathrm{d}x$. From $u = \sqrt{x-1}$ we get $x = u^2 - 1$ and $\mathrm{d}x = 2u\,\mathrm{d}u$. We can rewrite the integral as

$$\int x\sqrt{x-1}\,\mathrm{d}x = \int (u^2-1)u\cdot 2u\,\mathrm{d}u = \int 2u^4 - 2u^2\,\mathrm{d}u = \frac{2}{5}u^5 + \frac{2}{3}u^3 + C$$

and going back to x

$$\int x\sqrt{x-1}\,\mathrm{d}x = \frac{2}{5}(\sqrt{x-1})^5 + \frac{2}{3}(\sqrt{x-1})^3 + C$$

2. $\displaystyle\int 2x(x^2+1)^4\,\mathrm{d}x$. From $u = x^2 + 1$ we get $\mathrm{d}u = 2x\,\mathrm{d}x$. We can rewrite the integral as

$$\int 2x(x^2+1)^4\,\mathrm{d}x = \int u^4\,\mathrm{d}u = \frac{u^5}{5} + C = \frac{(x^2+1)^5}{5} + C$$

3. $\displaystyle\int \frac{(\ln x)^2}{x}\,\mathrm{d}x$. From $u = \ln x$ we get $x = e^u$ and $\mathrm{d}x = e^u\,\mathrm{d}u$. Thus

$$\int \frac{\ln^2 x}{x}\,\mathrm{d}x = \int \frac{u^2}{e^u}e^u\,\mathrm{d}u = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$$

4. $\displaystyle\int \frac{e^x}{e^x+2}\,\mathrm{d}x$. From $u = e^x$ we get $x = \ln u$ and $\mathrm{d}x = \mathrm{d}u/u$. Thus

$$\int \frac{e^x}{e^x+1}\,\mathrm{d}x = \int \frac{1}{u+2}\,\mathrm{d}u = \ln|u+2| + C = \ln|e^x+2| + C$$

14 1. From $u = \sqrt{x}$ we get $x = u^2$ and $\mathrm{d}x = 2u\,\mathrm{d}u$. The lower and upper limits of the x–integral are 1 and 4. The corresponding limits for the u–integral are $u(1) = \sqrt{1} = 1$ and $u(4) = \sqrt{4} = 2$. Thus

$$\int \frac{1}{u^2+u}2u\,\mathrm{d}u = 2\int \frac{1}{u+1}\,\mathrm{d}u = 2\ln|u+1| + C$$

and

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,\mathrm{d}x = 2\ln|u+1|\Big|_1^2 = 2\ln(3) - 2\ln(2)$$

2. $\displaystyle\int_1^e \frac{(\ln x)^2}{x}\,\mathrm{d}x$. From $u = \ln x$ we get $\mathrm{d}u = \mathrm{d}x/x$. Moreover, $u(1) = \ln 1 = 0$ and $u(e) = \ln e = 1$. Thus

$$\int_1^e \frac{(\ln x)^2}{x}\,\mathrm{d}x = \int_0^1 u^2\,\mathrm{d}u = \left.\frac{u^3}{3}\right|_0^1 = \frac{1}{3}$$

3. $\displaystyle\int_0^2 \frac{1}{(2x+3)^2}\,\mathrm{d}x$. From $u = 2x+3$ we get $\mathrm{d}u = 2\,\mathrm{d}x$. Moreover, $u(0) = 3$ and $u(2) = 7$. Thus

$$\int_0^2 \frac{1}{(2x+3)^2}\,\mathrm{d}x = \int_3^7 \frac{1}{u^2}\cdot\frac{\mathrm{d}x}{2} = \frac{1}{2}\left(-\frac{1}{u}\right)\Bigg|_3^7 = \frac{1}{2}\left(-\frac{1}{7}+\frac{1}{3}\right) = \frac{2}{21}$$

4. $\displaystyle\int_0^4 \frac{x}{x^2+1}\,\mathrm{d}x$. From $u = x^2+1$ we get $\mathrm{d}u = 2x\,\mathrm{d}x$ and $\mathrm{d}x = \mathrm{d}u/2$. Moreover, $u(0) = 1$ and $u(4) = 17$. Thus

$$\int_0^4 \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int_1^{17}\frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\ln u\Bigg|_1^{17} = \frac{\ln 17}{2}$$

15 Use integration by parts. Set $u = x$ and $\mathrm{d}v = e^{-\frac{1}{2}x}\,\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = -2e^{-\frac{1}{2}x}$. Therefore:

$$\int xe^{-\frac{1}{2}x}\,\mathrm{d}x = -2xe^{-\frac{1}{2}x} + \int 2e^{-\frac{1}{2}x}\,\mathrm{d}x = -2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} + c$$

The indefinite integral is $F(x) = -2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} + c$. The condition $F(0) = -1$ gives $F(0) = -2\cdot 0\cdot e^{-\frac{1}{2}0} - 4e^{-\frac{1}{2}0} + c = -4 + c = -1$ which yields $c = 3$.

16 Note that $e^{2x} = (e^x)^2$. Then

$$\int \frac{2e^x}{(e^x)^2-1}\,dx$$

Setting $u = e^x$ we get $\mathrm{d}u = e^x\,\mathrm{d}x$ and this yields

$$\int \frac{2e^x}{(e^x)^2-1}\,\mathrm{d}x = \int \frac{2}{u^2-1}\,\mathrm{d}u$$

The last integral is a rational function and can be solved writing the integrand as a sum:

$$\int \frac{2}{u^2-1}\,\mathrm{d}u = \int\left(\frac{1}{u-1} - \frac{1}{u+1}\right)\mathrm{d}u$$

Therefore

$$\int \frac{2}{u^2-1}\,\mathrm{d}u = \ln|u-1| - \ln|u+1| + C = \ln|e^x-1| - \ln(e^x+1) + C$$

17 From $y = 1+3x^2$ we get $\mathrm{d}y = 6x\,\mathrm{d}x$ so that $\mathrm{d}y/6 = x\,\mathrm{d}x$. The indefinite integral, i.e. the set of the antiderivatives of f, is

$$\int \frac{x}{\sqrt{3x^2+1}}\,\mathrm{d}x = \int \frac{1}{6\sqrt{y}}\,\mathrm{d}y = \frac{1}{6}\int y^{-1/2}\,\mathrm{d}y = \frac{1}{6}\left(\frac{y^{1/2}}{\frac{1}{2}}\right) + C$$
$$= \frac{1}{3}\sqrt{y} + C = \frac{1}{3}\sqrt{1+3x^2} + C$$

The lower and upper limits of the x–integral are 0 and 1. Thus, the lower and upper limits of the y–integral are $y(0) = 1$ and $y(1) = 4$. Therefore

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{3}\sqrt{y}\Big|_1^4 = \frac{1}{3}\sqrt{4} - \frac{1}{3}\sqrt{1} = \frac{1}{3}$$

18 1. $\int_3^{+\infty} x^{-2}\,\mathrm{d}x$. We have

$$\int_3^b x^{-2}\,\mathrm{d}x = -\frac{1}{x}\Big|_3^b = -\frac{1}{b} + \frac{1}{3}$$

and

$$\lim_{b\to+\infty}\left(-\frac{1}{b} + \frac{1}{3}\right) = \frac{1}{3}$$

thus

$$\int_3^{+\infty} x^{-2}\,\mathrm{d}x = \frac{1}{3}$$

2. $\int_0^{+\infty} \frac{1}{(x+1)^3}\,\mathrm{d}x$. Use substitution rule: $u = x+1$, $x = u-1$, $\mathrm{d}u = \mathrm{d}x$, $u(0) = 1$ and $u(b) = 1+b$. We have

$$\int_0^b \frac{1}{(x+1)^3}\,\mathrm{d}x = \int_1^{b+1} \frac{1}{u^3}\,\mathrm{d}u = -\frac{1}{2u^2}\Big|_1^{b+1} = -\frac{1}{2(b+1)^2} + \frac{1}{2}$$

and

$$\lim_{b\to+\infty}\left(-\frac{1}{2(b+1)^2} + \frac{1}{4}\right) = \frac{1}{2}$$

thus

$$\int_0^{+\infty} \frac{1}{(x+1)^3}\,\mathrm{d}x = \frac{1}{2}$$

3. $\int_1^{+\infty} e^{-x}\,\mathrm{d}x$. We have

$$\int_1^b e^{-x}\,\mathrm{d}x = -e^{-x}\Big|_1^b = -e^{-b} + e^{-1}$$

and

$$\lim_{b\to+\infty}\left(-e^{-b} + e^{-1}\right) = e^{-1}$$

thus

$$\int_1^{+\infty} e^{-x}\,\mathrm{d}x = e^{-1}$$

4. $\int_2^{+\infty} \frac{2}{\sqrt{x}}\,\mathrm{d}x$. We have

$$\int_2^b \frac{2}{\sqrt{x}}\,\mathrm{d}x = 4\sqrt{x}\Big|_2^b = 4\sqrt{b} - 4\sqrt{2}$$

and

$$\lim_{b\to+\infty}\left(4\sqrt{b} - 4\sqrt{2}\right) = +\infty$$

thus the integral diverges.

5. $\displaystyle\int_0^{+\infty} xe^{-x^2}\,\mathrm{d}x$. Rewrite the integral as

$$-\frac{1}{2}\int -2xe^{-x^2}\,\mathrm{d}x$$

and use pattern

$$\int f'e^f = e^f + C.$$

Then

$$\int_0^b xe^{-x^2}\,\mathrm{d}x = \left(-\frac{1}{2}(e^{-x^2})\right)\Bigg|_0^b = -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^0 = \frac{1}{2} - \frac{1}{2}e^{-b^2}$$

and

$$\lim_{b\to+\infty} \frac{1}{2} - \frac{1}{2}e^{-b^2} = \frac{1}{2}$$

thus

$$\int_0^{+\infty} xe^{-x^2}\,\mathrm{d}x = \frac{1}{2}$$

6. $\displaystyle\int_{-\infty}^0 e^{2x}\,\mathrm{d}x$ We have

$$\int_a^0 e^{2x}\,\mathrm{d}x = \frac{1}{2}e^{2x}\Bigg|_a^0 = \frac{1}{2}\left(e^0 - e^{2a}\right)$$

and

$$\lim_{a\to-\infty} \frac{1}{2}\left(e^0 - e^{2a}\right) = \frac{1}{2}$$

thus

$$\int_{-\infty}^0 e^{2x}\,\mathrm{d}x = \frac{1}{2}$$

7. $\displaystyle\int_{-\infty}^0 e^{3x}\,\mathrm{d}x$. We have

$$\int_a^0 e^{3x}\,\mathrm{d}x = \frac{1}{3}e^{2x}\Bigg|_a^0 = \frac{1}{3}\left(e^0 - e^{3a}\right)$$

and

$$\lim_{a\to-\infty} \frac{1}{3}\left(e^0 - e^{3a}\right) = 1$$

thus

$$\int_{-\infty}^0 e^{3x}\,\mathrm{d}x = \frac{1}{3}$$

19 1. $\displaystyle\int_3^{+\infty} \frac{2x}{5x^3+x^2+1}\,\mathrm{d}x$. Since

$$\frac{2x}{5x^3+x^2+1} \sim \frac{2x}{5x^3} = \frac{2}{5}\frac{1}{x^2}$$

and we know (see theorem 8) that

$$\int_3^{+\infty} \frac{1}{x^2}\,\mathrm{d}x \qquad \text{converges}$$

then

$$\int_3^{+\infty} \frac{x^2}{x^3+x^2+1}\,\mathrm{d}x \qquad \text{converges}$$

2. $\displaystyle\int_3^{+\infty} \frac{3x^3+\sqrt{x}}{2x^3+x^2+\sqrt{x}}\,\mathrm{d}x$. Since

$$\frac{3x^3+\sqrt{x}}{2x^3+x^2+\sqrt{x}} \sim \frac{3x^3}{2x^3} = \frac{3}{2}$$

and we know that

$$\int_3^{+\infty} \frac{3}{2}\,\mathrm{d}x \qquad \text{does not converge,}$$

then

$$\int_3^{+\infty} \frac{x^2}{x^3+x^2+1}\,\mathrm{d}x \qquad \text{does not converge}$$

3. $\displaystyle\int_3^{+\infty} \frac{1}{x\sqrt{x+1}}\,\mathrm{d}x$. Since

$$\frac{1}{x\sqrt{x+1}} \sim \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$$

and we know (see theorem 8) that

$$\int_3^{+\infty} \frac{1}{x^{3/2}}\,\mathrm{d}x \qquad \text{converges}$$

then

$$\int_3^{+\infty} \frac{1}{x\sqrt{x+1}}\,\mathrm{d}x \qquad \text{converges}$$

4. $\displaystyle\int_3^{+\infty} \frac{x^2}{x^3+1}\,\mathrm{d}x$. Since

$$\frac{x^2}{x^3+1} \sim \frac{1}{x}$$

and we know (see theorem 8) that

$$\int_3^{+\infty} \frac{1}{x}\,\mathrm{d}x \qquad \text{does not converge}$$

then

$$\int_3^{+\infty} \frac{x^2}{x^3+1}\,\mathrm{d}x \qquad \text{does not converge}$$

5. $\displaystyle\int_3^{+\infty} \frac{\ln x}{x^3}\,\mathrm{d}x$. Since $\ln x < x$ for $x > 3$ then

$$\frac{\ln x}{x^3} < \frac{x}{x^3} = \frac{1}{x^2}$$

and we know (see theorem 8) that

$$\int_3^{+\infty} \frac{1}{x^2}\,\mathrm{d}x \qquad \text{converges}$$

then

$$\int_3^{+\infty} \frac{\ln x}{x^3}\,\mathrm{d}x \qquad \text{converges}$$

6. $\int_1^{+\infty} x^3 e^{-x}\,\mathrm{d}x$. Since $e^x > x^5$ as $x \to +\infty$ (actually, $e^x > x^p$ for any p as $x \to +\infty$) then

$$x^3 e^{-x} = \frac{x^3}{e^x} < \frac{x^3}{x^5} = \frac{1}{x^2}$$

and we know (see theorem 8) that

$$\int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x \qquad \text{converges}$$

then

$$\int_1^{+\infty} x^3 e^{-x}\,\mathrm{d}x \qquad \text{converges}$$

20 The indefinite integral of f is

$$F(x) = -2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} + c.$$

Thus

$$\int_0^k f(x)\,\mathrm{d}x = F(k) - F(0) = -2ke^{-\frac{1}{2}k} - 4e^{-\frac{1}{2}k} - (0-4)$$

and finally

$$\lim_{k\to+\infty}\left(-2ke^{-\frac{1}{2}k} - 4e^{-\frac{1}{2}k} + 4\right) = 4$$

21 1. The graph of the derivative of $F(x)$ is that of $f(x)$ (see Figure 5, left). $F(x)$ is an increasing function and $F(0) = 0$. On $[0,1]$, we have

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x t\,\mathrm{d}t = \left.\frac{t^2}{2}\right|_0^x = \frac{x^2}{2}$$

If $1 \le x \le 2$ then

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t + \int_1^x f(t)\,\mathrm{d}t = \left.\frac{t^2}{2}\right|_0^1 + \int_1^x (t-1)\,\mathrm{d}t = \frac{1}{2} + \left.\frac{(t-1)^2}{2}\right|_1^x = \frac{1}{2} + \frac{(x-1)^2}{2} = \frac{x^2}{2} - x + 1$$

Therefore,

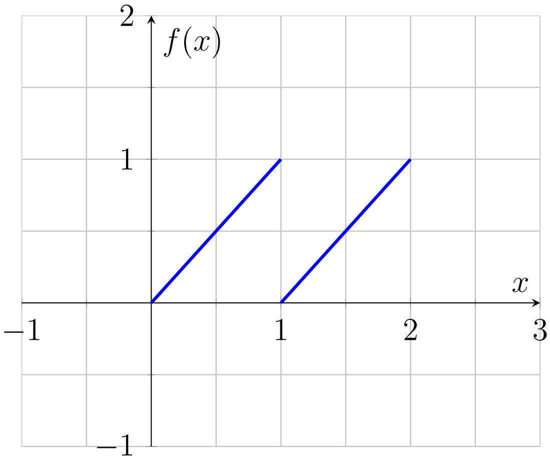
$$F(x) = \begin{cases} \dfrac{x^2}{2} & 0 \le t \le 1 \\ \dfrac{x^2}{2} - x + 1 & 1 < x \le 2 \end{cases}$$

2. Since f is piecewise continuous it is integrable and $F(x)$ is continuous. Moreover, $F(0) = 0$. For $x \le 0$ we have

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x 0\,\mathrm{d}t = 0$$

If $x \in [0,5]$ we have

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x 5\,\mathrm{d}t = 5t\Big|_0^x = 5x$$



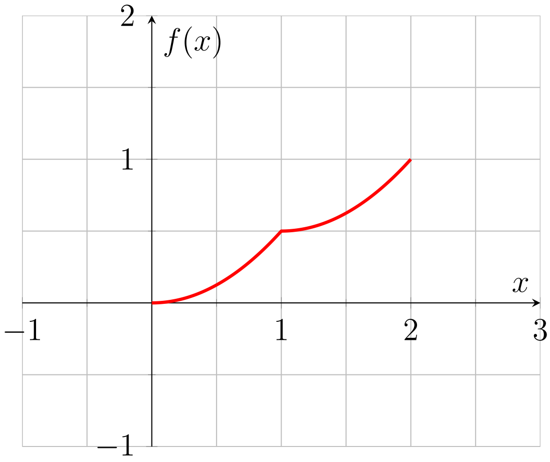


Figure 5: $f(x)$ (on the left) and its integral function $F(x) = \int_0^x f(t)\,dt$ on the right.

If $x > 5$

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x 5\,\mathrm{d}t + \int_5^x 15\,\mathrm{d}t = 25 + 15t\Big|_5^x = 25 + 15x - 75 = 15x - 50$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 0 \\ 5x & 0 \le x \le 5 \\ 15x - 50 & x > 5 \end{cases}$$

3. Since f is continuous it is also integrable and $F(x)$ is differentiable and $F'(x) = f(x)$. Moreover, $F(0) = 0$. For $x \le 0$ we have

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x e^t\,\mathrm{d}t = -\int_x^0 e^t\,\mathrm{d}t = -e^t\Big|_x^0 = -(e^0 - e^x) = e^x - 1$$

If $x > 0$ we have

$$F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x e^{-t}\,\mathrm{d}t = -e^{-t}\Big|_0^x = e^0 - e^{-x} = 1 - e^{-x}$$

Therefore (see figure 9),

$$F(x) = \begin{cases} e^x - 1 & x \le 0 \\ 1 - e^{-x} & x > 0 \end{cases}$$

4. Since f is continuous it is also integrable and $F(x)$ is differentiable and $F'(x) = f(x)$. Moreover, $\lim_{x \to -\infty} F(x) = 0$. For $x \le 0$ we have

$$F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t = \int_{-\infty}^x e^t\,\mathrm{d}t = \int_{-\infty}^x e^t\,\mathrm{d}t = e^x$$

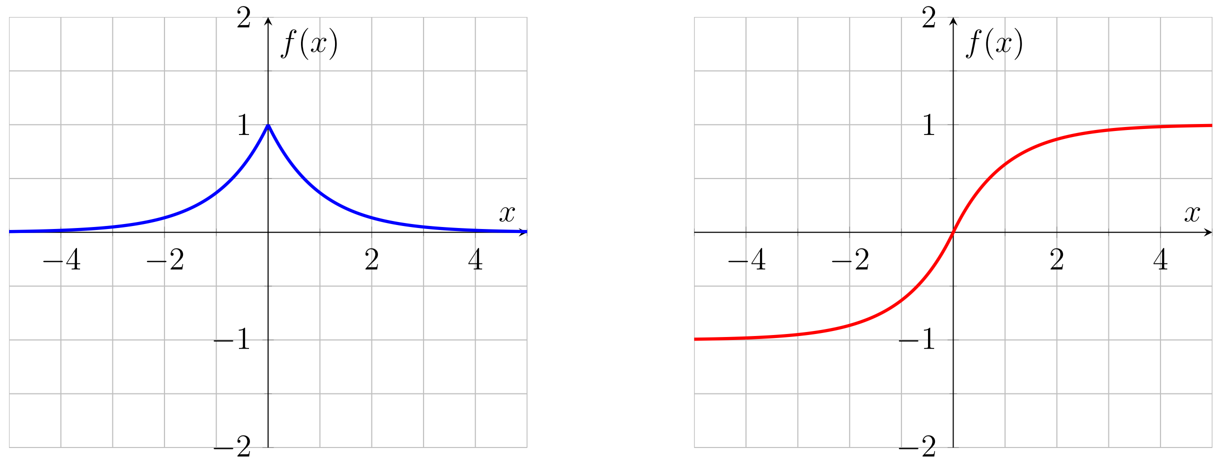


Figure 6: $f(x)$ (on the left) and its integral function $F(x) = \int_0^x f(t)\,\mathrm{d}t$ on the right.

If $x > 0$ we have

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{0} e^t\,\mathrm{d}t + \int_{0}^{x} e^{-t}\,\mathrm{d}t = 1 + \left(-e^{-t}\Big|_0^x\right) = 1 + \left(-e^{-x}+1\right) = 2 - e^{-x}$$

Therefore (see figure 7),

$$F(x) = \begin{cases} e^x & x \leq 0 \\ 2 - e^{-x} & x > 0 \end{cases}$$

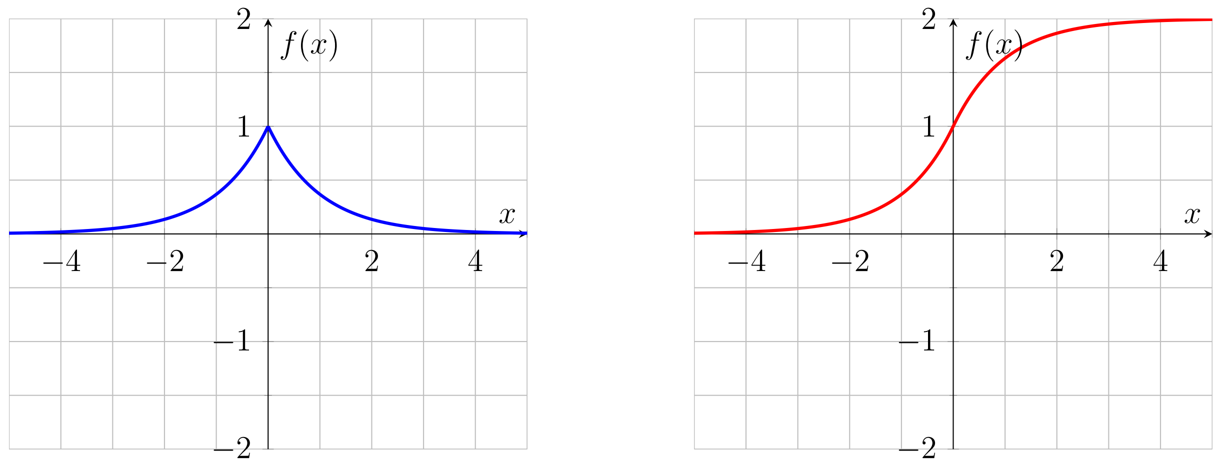


Figure 7: $f(x)$ (on the left) and its integral function $F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t$ on the right.

5. Since f is continuous it is also integrable and $F(x)$ is differentiable and $F'(x) = f(x)$. Moreover, $\lim_{x\to-\infty} F(x) = 0$. For $x \leq -1$ we have

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{x} \frac{1}{t^2}\,\mathrm{d}t = -\frac{1}{x}$$

If $-1 < x < 1$ we have

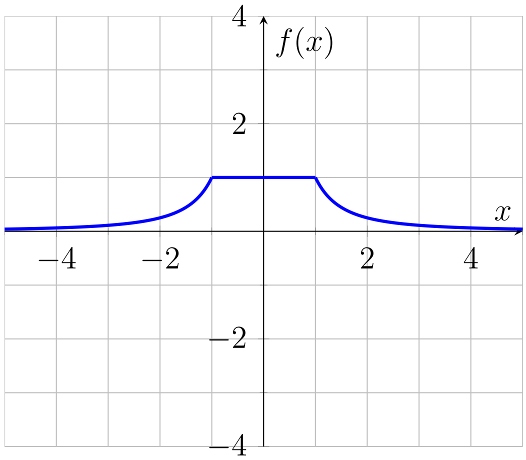
$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{-1} \frac{1}{t^2}\,\mathrm{d}t + \int_{-1}^{x} 1\,\mathrm{d}t = 1 + \left(t\Big|_{-1}^{x}\right) = 1 + (x+1) = 2 + x$$

If $x \geq 1$ we have

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{-1} \frac{1}{t^2}\,\mathrm{d}t + \int_{-1}^{1} 1\,\mathrm{d}t + \int_{1}^{x} \frac{1}{t^2}\,\mathrm{d}t = 1 + 2 - \left(\frac{1}{t}\Big|_1^x \right) = 4 - \frac{1}{x}$$

Therefore (see figure 8),

$$F(x) = \begin{cases} -\dfrac{1}{x} & x \leq 1 \\ 2 + x & -1 < x < 1 \\ 4 - \dfrac{1}{x} & x \leq 1 \end{cases}$$



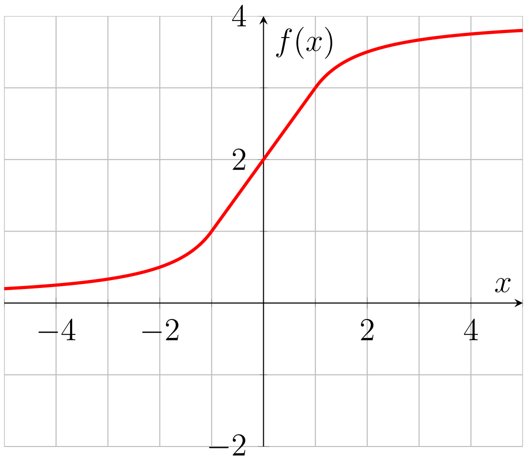


Figure 8: $f(x)$ (on the left) and its integral function $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ on the right.

22 b).

23 a).

24 Since f is continuous we can apply the mean value theorem. The mean value of f is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{1}{b-a} \cdot 0 = 0$$

and by the theorem there is at least one number c such that $f(c)$ equals the mean value, that is $f(c) = 0$.

Continuity is indeed necessary. If $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$ then for every $b > 0$ we have

$$\int_{-b}^{b} f(x)\,\mathrm{d}x = 0$$

even if $f(x) \neq 0$ for all $x \in \mathbb{R}$.

25 Say c is a continuity point for f and suppose $f(c) > 0$. Then, by continuity, for every $\varepsilon > 0$ there is a δ such that if $0 < |x - c| < \delta$ then $\big|f(x) - f(c)\big| < \varepsilon$. Take $\varepsilon = f(c)/2$ and let δ_0 the corresponding δ. Then

$$\big|f(x) - f(c)\big| < \frac{f(c)}{2} \qquad |x - c| < \delta_0$$

that is

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2} \qquad |x-c| < \delta_0$$

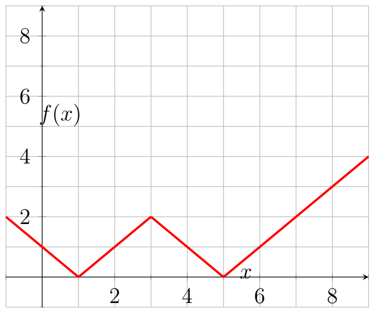
Thus, since f is non-negative, we have

$$\int_a^b f(x)\,\mathrm{d}x \geq \int_{c-\delta_0}^{c+\delta_0} f(x)\,\mathrm{d}x > \frac{f(c)}{2}\int_{c-\delta_0}^{c+\delta_0} \mathrm{d}x = f(c)\delta_0 > 0$$

which contradicts the hypothesis about the integral of f. Then, it must be that $f(x) = 0$ at every point of continuity.

Note that the function need not be 0 at isolated points. For example, the previous function can be equal to 1 at a point in $[a,b]$, say $(a+b)/2$. The integral would be 0 anyway, since the value of f at any single point does not influence that of the integral.

26 $f(x) \geq 0$ for every x so the integral equals the area under the curve. The graph of f is



and therefore

$$\int_0^8 f(x)\,\mathrm{d}x = \frac{1}{2} + 4 + \frac{9}{2} = 9$$

27 $f(x) = e^{x^2}$ is strictly increasing on $[0,1]$ and $f(0) = 1$ and $f(1) = e$. Thus we immediately have

$$1 \leq \int_0^1 e^{x^2}\,\mathrm{d}x.$$

To prove the second inequality, note that $f'(x) = 2xe^{x^2}$ and

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = 2e^{x^2}(1+2x2) > 0$$

therefore, f is convex on $[0,1]$ which means it's bounded by the chord passing through $(0,1)$ and $(1,e)$. Then the area under f is bounded by that of the trapezoid with sides 1, e and height 1, that is $(1+e)/2$.

28 The slope of the tangent line to F is the value of $F'(x) = f(x)$. At $x = 3$ this is equal to

$$F'(3) = f(3) = \frac{4}{9-4} = \frac{4}{5}$$

and the equation of the tangent line to F in $x_0 = 3$ is

$$\begin{aligned} y &= F(3) + F'(3)(x-3) \\ &= 0 + f(3)(x-3) \\ &= \frac{4}{5}(x-3) \end{aligned}$$

29 The integrand is defined for any real t because the denominator never vanishes. Thus F is defined for all real x. The two improper integrals

$$\int_{-\infty}^{1} \frac{t+1}{t^2+1}\ \mathrm{d}t \qquad \int_{1}^{+\infty} \frac{t+1}{t^2+1}\ \mathrm{d}t$$

both diverge and therefore

$$\lim_{x\to-\infty} F(x) = +\infty$$

and

$$\lim_{x\to+\infty} F(x) = +\infty.$$

We have

$$F'(x) = \frac{x+1}{x^2+1}$$

and

$$F''(x) = \frac{-x^2-2x+1}{(x^2+1)^2}$$

whence we see that $x = -1$ is a stationary point. Since $F''(-1) = 1/2 > 0$, F has a local (and global) point of minimum at $x = -1$. F'' is positive only in $-1-\sqrt{2} < x < -1+\sqrt{2}$ and thus the function is convex in that interval and concave outside of it. Finally, $F(1) = 0$ (see figure 9).

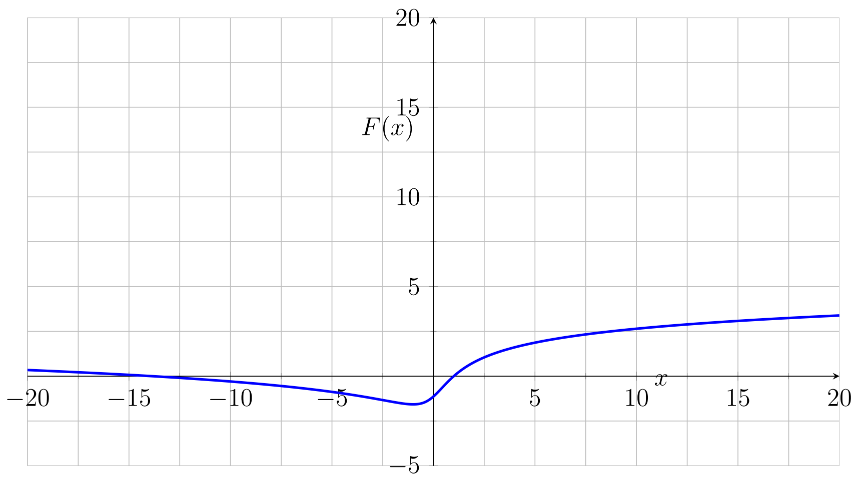


Figure 9: $F(x) = \displaystyle\int_{1}^{x} \frac{t+1}{t^2+1}\ \mathrm{d}t$

The definite integral can actually be computed using techniques that are beyond the scope of these lectures. One finds that

$$F(x) = \int_{1}^{x} \frac{t+1}{t^2+1}\ \mathrm{d}t = \frac{1}{2}\ln(x^2+1) + \arctan x - \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

30 Let

$$f(t) = t^4 e^{-t^2}.$$

Since $f(t) = f(-t)$ and $f(t)$ is continuous, F is differentiable and it's an even function (that is, $F(-x) = -F(x)$). The improper integral

$$\int_{0}^{+\infty} f(t)\ \mathrm{d}t$$

converges, say to $a > 0$. Thus

$$\lim_{x\to+\infty} F(x) = a$$

and by symmetry

$$\lim_{x\to-\infty} F(x) = -a$$

We have

$$F'(x) = x^4 e^{-x^2}$$

which is non–negative for all x (and vanishes only if $x = 0$). In addition

$$F''(x) = -2e^{-x^2} x^3 (x^2 - 2)$$

which is positive if $x < -\sqrt{2}$ or $0 < x < \sqrt{2}$. In these intervals, F is convex. For $x > \sqrt{2}$ or $-\sqrt{2} < x < 0$, F is concave. The graph of F is sketched in figure 10.

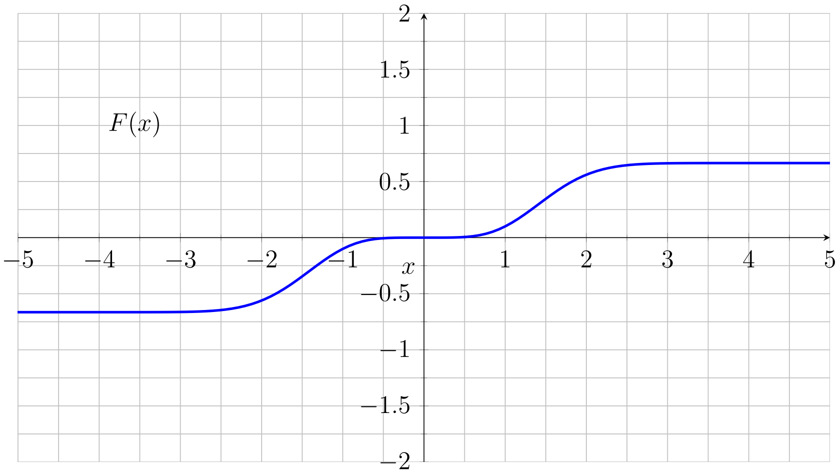


Figure 10: $F(x) = \displaystyle\int_0^x t^4 e^{-t^2}\,\mathrm{d}t$

31 The two functions

$$g(x) = \int_0^x f(t)\,\mathrm{d}t$$

and

$$h(x) = \int_x^1 f(t)\,\mathrm{d}t = -\int_1^x f(t)\,\mathrm{d}t$$

are both differentiable, since f is continuous, and their derivatives are

$$g'(x) = f(x), \qquad h'(x) = -f(x).$$

Then, differentiating the equality

$$\int_0^x f(t)\,\mathrm{d}t = \int_x^1 f(t)\,\mathrm{d}t, \qquad \forall x \in [0,1]$$

we get

$$f(x) = -f(x), \qquad \forall x \in [0,1]$$

which means $f(x) = 0$ for all $x \in [0,1]$.

32 a).

33 c).

34 *a*) true

b) true

c) false

d) false

e) true

35 *a*) true

b) false

c) false

d) false

36 *a*) false

b) true

c) false

d) false

e) false

f) true

37 b).

38 c).

39 a).