

COMPUTATIONAL MICROECONOMICS

(GAME THEORY)



Rationality, Preferences and Utility

"Economic theory studies interaction between individuals (**Equilibrium**). But let's start with individual decision making (**Rationality**)".

Individual Decision Making

Human, or even animals and plants, solve decision problems all the time by selecting alternatives among the available ones.

Rational choice theory (the first half of the course) assumes that they do it purposefully and optimally (rationality).

Rationality

Rationality is in economics a property that people's choices may, or may not have. Rationality in everyday language. Logical/make sense?

In economics, choices are "rational" if they are best according to the decision maker's **preferences**. But what those preferences are is left to the decision maker. Everyone can have their own preferences. Economics is not judgemental.

What is preference? In economic theory, an individual is ranking of the alternatives, not about the intensity of his feelings. Economists use binary relation to encode the answers.

Preferences: Binary Relation

A binary relation R on a set X specifies, for each ordered pair (x, y) of members of X , whether or not x relates to y ($x R y$) in a certain way. We encode the answer to it by $\geq$. $x \geq y$: x is at least as desirable as y .

"I prefer x to y ": $x \geq y$ but not $y \geq x$. "I regard x and y as equally desirable": $x \geq y$ and $y \geq x$ or $x \sim y$

Definition 1-1: Reflexive binary relation

A binary relation R on the set X is **reflexive** if for all member x of X , $x R x$.

Definition 1-2: Complete binary relation

A binary relation R on the set X is **complete** if for all members x and y of X , either $x R y$ or $y R x$ (or both).

Question: Is a complete binary relation reflexive? Yes, it is.

Definition 1-3: Transitive binary relation

A binary relation R on the set X is transitive if for any member x, y and z of X for which $x R y$ and $y R z$, we have $x R z$.

Definition 1-4: Preference relation

A **preference relation** on the set X is a complete and transitive binary relation on X .

Exercise: Encode answers by the binary relation $\geq$ on $X = \{\text{Old Fashioned, Margarita, Mojito}\}$ or $Y = \{\text{Fanta, Coca, Pepsi}\}$.

Do the answers consist of a preference relation? Yes, if we have completeness and transitivity.

Definition 1-5: Symmetric and Antisymmetric binary relations

A binary relation R on the set X is **symmetric** if for any members x and y of X for which $x R y$ we have $y R x$, and is **antisymmetric** if for any members x and y of X for which $x \neq y$ and $x R y$, it is not the case that $y R x$.

- Could you give me an example of symmetric binary relation? Equality
- Could you give me an example of antisymmetric binary relation? "Paid the restaurant bill of ..."
- Is there a binary relation that is symmetric and antisymmetric? Yes, in this case is coreflexive, ex. an empty relation
- Is there a binary relation that is neither symmetric nor antisymmetric? "Less-than-or-Equal" ($\leq$) No Symmetry and No Antisymmetry: $5 \leq 7 \rightarrow 7 \not\leq 5$.

Definition 1-6: Equivalence Relation

A binary relation is an **equivalence** relation if it is reflexive, symmetric, and transitive.

Any equivalence relation R on a set X divides X into disjoint subsets such that two alternatives x and y belong to the same subset if and only if $x R y$. Each of these subsets is called an equivalence class. For the indifference relation, the equivalence classes are referred to also as indifference sets; the individual regards all alternatives in an indifference set as equally desirable and alternatives in different indifference sets as not equally desirable.

Utility

You may have heard, perhaps in an introductory economics class, that economics studies the "utility maximizing" agent.

Indeed, in many economic models, an individual is described not by his preferences but by a value function. What is utility? Why do we use it?

A preference is a complicated object. If $|X| = 100$, there are 4950 pairs. The list will become very long! An easier way of writing down preferences is to ask one to assign each item a number. The higher the number the more she prefers an item. $u(a) = 5, u(b) = 2$ implies $a \succ b$.

If we are able to write down one's preferences by this number assigning way, we say her preferences can be represented by a utility function.

Definition 1-7: Utility function

For any set X and preference relation $\succsim$ on X , the function $u: X \rightarrow \mathbb{R}$ represents $\succsim$ if $x \succsim y$ and if $u(x) \geq u(y)$.

We say u is a **utility function** for $\succsim$.

A natural question: which preference relation can be represented by a utility function? Every preference relation on a **finite set** can be represented by a utility function.

Definition 1-8: Minimal and maximal alternatives

For any set X and preference relation $\succsim$ on X , the alternative $x \in X$ is minimal with respect to $\succsim$ in X if $y \succsim x$ for all $y \in X$ and is maximal with respect to $\succsim$ in X if $x \succsim y$ for all $y \in X$.

Every preference relation on a finite set has minimal and maximal members.

Lemma 1-1: Existence of minimal and maximal alternatives

Let X be a nonempty finite set and let $\succsim$ be a preference relation on X . At least one member of X is minimal with respect to $\succsim$ in X and at least one member is maximal.

Proof: We prove the result for maximality by induction.

Exam

- Suppose $|X| = 1$. The single element is maximal.
- Suppose the lemma is true for $|X| = n-1$. We prove it is true for $|X| = n$.
- Pick any element $y \in X$. $X \setminus \{y\}$ has $n-1$ elements.
- There is $x \in X \setminus \{y\}$ that is maximal.
- If $x \succsim y$, then x is maximal w.r.t. $\succsim$ in X .
- If not, then $y \succ x$. Because $x \succsim z$ for any $z \in X \setminus \{y\}$, by transitivity $y \succsim z$. Q.E.D.

Proof: (Def. 1-8)
in class

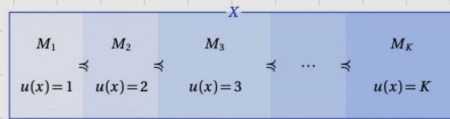
$|X| = 1, |X| = n-1, |X| = n \xrightarrow{\text{Pick } y \in X} |X \setminus \{y\}| = n-1$ then $\exists x \in X \setminus \{y\}$ s.t. $\forall z \in X \setminus \{y\}$ and $x \succsim z$.
↳ It has a maximum element

① $x \succsim y$
We can compare x and y ② $y \succ x$ then y is the maximum element thanks to transitivity

Proposition 1-1: Representing preference relation by utility function

Every preference relation on a finite set can be represented by a utility function.

Proof: Let X be a finite set and $\succsim$ be a preference relation on X . Let $Y_0 = X$ and M_1 be the set of alternatives minimal w.r.t. $\succsim$ in Y_0 .
 Lemma implies M_1 is nonempty. Inductively, $Y_k = Y_{k-1} \setminus M_k$ and M_{k+1} be the set of alternatives minimal w.r.t. $\succsim$ in Y_k .



Let $u(x) = k$ for all $x \in M_k$, $k = 1, 2, \dots, K$. WWTs $a \succsim b$ iff $u(a) \geq u(b)$

Case 1: $u(a) = u(b)$ iff they are both minimal in $Y_{u(a)-1}$. So that $a \succsim b$ and $b \succsim a$. Hence $a \sim b$.

Case 2: $u(a) > u(b)$ iff b is minimal w.r.t. $\succsim$ in $Y_{u(b)-1}$, so that $a \succ b$, and $a \in Y_{u(b)-1}$ is not minimal.

So that it is not the case $b \succ a$. Hence, $a \succ b$. Q.E.D.

Exam

Many preference relations on infinite sets can also be represented by utility functions. $X = [0, 1]$, $x \succsim y$ iff $x \geq y$.

$u(x) = x$ represents $\succsim$. However, not all preference relations on infinite sets can be represented by utility functions.

A famous counter-example:

Lexicographic preference relation over the unit square $X = \{(x_1, x_2) : x_1, x_2 \in [0, 1]\}$ for which the first priority is the first coordinate and the second priority is the second coordinate e.g., $(0.3, 0.1) \succ (0.2, 0.9) \succ (0.2, 0.3)$.

Proposition 1-2: preference relation not represented by utility function

The (lexicographic) preference relation $\succsim$ on $X = \{(x_1, x_2) : x_1, x_2 \in [0, 1]\}$ defined by $(x_1, x_2) \succ (y_1, y_2)$ if and only if either: (i) $x_1 > y_1$ or

(ii) $x_1 = y_1$ and $x_2 > y_2$ is not represented by a utility function.

(Non) Uniqueness

If a preference relation can be represented by one utility function, can it be represented by other utility functions?

Proposition 1-3: Increasing function of utility function is utility function

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function. If u represents the preference relation $\succsim$ on X , then so does the function w defined by $w(x) = f(u(x))$ for all $x \in X$.

Proof: $w(x) \geq w(y)$ iff $f(u(x)) \geq f(u(y))$ iff $u(x) \geq u(y)$ iff $x \succsim y$.

Exam

What do utility level mean?

Hedonic Interpretation: Utility is an objective measure of individual's well being.

"Nature has placed mankind under the governance of two sovereign masters, pain and pleasure. It is for them alone to point out what we ought to do... By the principle of utility is meant that principle which approves or disapproves of every action whatsoever according to the tendency it appears to have to augment or diminish the happiness of the party whose interest is in question: or, what is the same thing in other words to promote or to oppose that happiness. I say of every action whatsoever, and therefore not only of every action of a private individual, but of every measure of government" - Jeremy Bentham.

Choice

- Hypothetical questions vs. Real Choice questions.
- Stated preferences vs. Revealed preferences.
- A preference relation refers only to the individual's mental attitude, not to the choices he/she may make.
- In this lecture, we describe a concept of choice, independently of preferences.

Choice and Rational Choice

Definition 2-1: Choice problem and Choice function

Given a set X , a choice problem for X is a nonempty subset of X and a choice function for X , $c: 2^X \setminus \emptyset \rightarrow X$, associates with every choice problem $A \subseteq X$ a single member of A (the member chosen).

Rational choice: choices are "rational" if they are best according to the decision maker's preferences.

- Individual has a preference relation $\succeq$ over X .
- Whenever she has to make a choice, she is aware of the set of possible alternatives.
- She chooses an alternative that is best according to her preference over the set of possible alternatives.

Remark 1: Recall economics is not judgemental, the individual's preferences can be "irrational" in everyday language.

Remark 2: As an outsider, we do not know $\succeq$.

If the preference relation of an individual is represented by the utility function u . Then the individual acts as if he maximizes the function u under the constraint that $x \in A$. $\max \{u(x) : x \in A\}$.

Rationalizing choice: An *if* argument

Human beings usually do not consciously maximize a preference relation when they make decisions. Although individuals rarely explicitly choose the best alternative according to their preference relation. The standard justification for modelling individuals as rational is that their behaviour can often be described *as if* they make choices in this way. Individuals do not have to be aware of their preference relations. The assumption that they maximize some preference relation is appropriate as long as we can describe them *as if* they behave in this way.

"Consider the problem of predicting the shots made by an expert billiard player. It seems not at all unreasonable that excellent predictions would be yielded by the hypothesis that the billiard player made his shots as if he knew the complicated mathematical formulas that would give the optimum directions of travel, could estimate accurately by eye the angles, etc., describing the location of the balls, could make lightning calculations from the formulas, and could then make the balls travel in the direction indicated by the formulas. Our confidence in this hypothesis is not based on the belief that billiard players, even expert ones, can or do go through the process described; it derives rather from the belief that, unless in some way or other they were capable of reaching essentially the same result, they would not in fact be expert billiard players." - Milton Friedman, Essays in Positive Economics, 1953.

Rationalizing Choice: Definition

Definition 2-2: Rationalizable choice function

A choice function is **rationalizable** if there is a preference relation such that for every choice problem the alternative specified by the choice function is the best alternative according to the preference relation.

Example: Let $X = \{a, b, c\}$.

- $c(X) = a$, $c(\{a, b\}) = a$, $c(\{a, c\}) = a$, $c(\{b, c\}) = b$. Is $c(\cdot)$ rationalizable? Yes, it is.
- $c(X) = a$, $c(\{a, b\}) = a$, $c(\{a, c\}) = c$, $c(\{b, c\}) = c$. Is $c(\cdot)$ rationalizable? No, it's not.

Definition 2-3: Property α

Given a set X , a choice function c for X satisfies property α if for any sets A and B with $B \subset A \subseteq X$ and $c(A) \in B$ we have $c(B) = c(A)$.

Whenever the choice from A is in a subset B then the alternative chosen from A is chosen also from B . Eliminating some of the unchosen alternatives from A shouldn't affect the selection of $c(A)$ as the best option. (Independence of irrelevant alternatives (IIA)).

"Property α is a sort of **mathematical consistency**". - Tangren Feng

Proposition 2-1: Rationalizable choice function satisfying property α .

Every rationalizable choice function satisfies property α .

- Proof:**
- Let c be a rationalizable choice function for X and let $\succsim$ be a preference relation that rationalizes c .
 - For any set $A \subseteq X$, $c(A) \succsim y$ for all $y \in A$.
 - Assume $B \subset A$. Then $c(A) \succsim y$ for all $y \in B$.
 - By assumption, $c(A) \in B$. Thus, $c(A) = c(B)$.

Exam

Proposition 2-2: Choice function satisfying property α is rationalizable

If X is a finite set then any choice function for X satisfying property α is rationalizable.

- Proof:**
- Let c be a choice function for X satisfying property α . Denote $|X|$ by n .
 - We construct a preference relation that rationalize c as follows.
 - Denote $c(X) = a_1$, $c(X \setminus \{a_1\}) = a_2$, $c(X \setminus \{a_1, a_2\}) = a_3$, and so on.
 - Consider $\succsim$ defined by $a_1 \succ a_2 \succ \dots \succ a_n$.
 - WWTs for all $A \subseteq X$, the maximal element in A w.r.t. $\succsim$ is $c(A)$.
 - Suppose the maximal element in A is a_m . Then any element in A ranked (weakly) lower than a_m , i.e., $A \subseteq \{a_m, a_{m+1}, \dots, a_n\}$.
 - We know $c(\{a_m, a_{m+1}, \dots, a_n\}) = a_m$.
 - By property α , $c(A) = a_m$.

Exam

The Money Pump argument

Economists often assume that a choice function is rationalizable. Why?

- Suppose $X = \{a, b, c\}$, $c(\{a, b\}) = a$, $c(\{a, c\}) = c$, $c(\{b, c\}) = b$.
- And suppose you have a at hand.
- Step 1. I offer to replace a with c in return for $1 \in$.
- Step 2. I offer to replace c with b in return for $1 \in$.
- Step 3. I offer to replace b with a in return for $1 \in$.
- Now you have a at hand and are $3 \in$ poorer.
- Repeat step 1-3 many times.

Rationalizing choice: Property α (Counter-Example)

Each of the people in the set $X = \{A, B1, B2\}$ organizes a party. A person might be invited to a subset of those parties and can attend only one party. Individuals $B1$ and $B2$ are both good friends of the partygoer but the relations between $B1$ and $B2$ are tense. The person's behaviour is as follows

- If she is invited by A and $B1$, she accepts $B1$'s invitation.
 - If she is invited by all three individuals, she accepts A 's invitation.
- * She does so because she is worried that accepting the invitation of $B1$ or $B2$ will be interpreted negatively by the other.

Remark 1: Obviously, such behaviour is not rationalizable by a preference relation over X .

Remark 2: The meaning of choosing one alternative ($B1$) is affected by the presence or absence of another alternative ($B2$).

Preferences Under Uncertainty

In the first topic, we discussed a model of preferences over an arbitrary set of alternatives. There are lots of instances in real life in which an alternative in the set involves randomness regarding the consequence it yields (lottery): Stocks, Education, Dinners in a new restaurant.

We study **preferences under uncertainty** in this topic.

Lotteries

What is a lottery in everyday language? A raffle ticket that yields a Ferrari with probability 0.001 and nothing otherwise.

A drawing of lots in which **prizes** are distributed to the winners among persons buying a **chance**.

Definition 3-1: Lotteries

Let Z be a set (of prizes). A lottery over Z is a function $p: Z \rightarrow \mathbb{R}$ that assigns a positive number (probability) $p(z)$ to a finite number of members of Z and 0 to all other members, with $\sum_{z \in Z} p(z) = 1$.

The support of the lottery p , denoted $\text{supp}(p)$, is the set of all prizes to which p assigns positive probability, $\{z \in Z : p(z) > 0\}$.

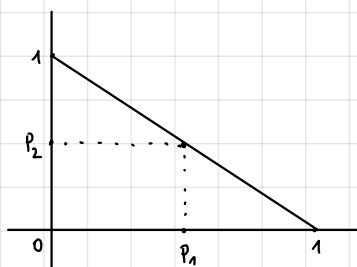
We denote the set of all lotteries over Z by $L(Z)$, the lottery that yields the prize z with probability 1 by $[z]$ (degenerate lottery), and the lottery that yields the prize z_k with probability α_k for $k=1, \dots, K$ by $\alpha_1 \cdot z_1 \oplus \alpha_2 \cdot z_2 \oplus \dots \alpha_K \cdot z_K$ or $(\alpha_1, \alpha_2, \dots, \alpha_K)$.

Examples: If Z consists of two prizes, z_1 and z_2 , then each member p of $L(Z)$ is specified by a pair (p_1, p_2)

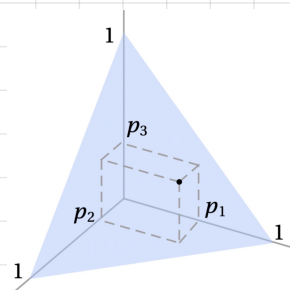
of nonnegative numbers with sum 1, where $p_1 = p(z_1)$ and $p_2 = p(z_2)$ are the probability of the prizes

Let $Z = \{\text{Ferrari}, \emptyset\}$. How to write a lottery that yields Ferrari with probability 0.5 and nothing with probability 0.5?

What is $L(Z)$?



The set $L(\{z_1, z_2\})$ of lotteries. The point (p_1, p_2) on the line segment represents the lottery p for which $p(z_1) = p_1$ and $p(z_2) = p_2$.



The set $L(\{z_1, z_2, z_3\})$ of lotteries. The point (p_1, p_2, p_3) in the triangle represents the lottery p for which $p(z_1) = p_1$, $p(z_2) = p_2$ and $p(z_3) = p_3$.

Preferences over lotteries

We are interested in preference relations over $L(Z)$. In terms of the model in Topic 1, the set X is equal to $L(Z)$.

Example 3.1: A pessimist

An individual has a strict preference relation $\succ^*$ over the set Z of prizes and (pessimistically) evaluates lotteries by the worst prize, according to the preference relation, that occurs with positive probability. That is, he prefers the lottery $p \in L(Z)$ to the lottery $q \in L(Z)$ if he prefers the worst prize that occurs with positive probability in p to the worst prize that occurs with positive probability in q .

- Let $Z = \{\$100, \$10, \$1\}$ and $\$100 \succ^* \$10 \succ^* \$1$.
- What is a pessimist's preference over $p = (0.9, 0, 0.1)$ and $q = (0.5, 0.5, 0)$? He prefers q .

Example 3.2: Good and Bad

An individual divides the set Z of prizes into two subsets, good and bad. For any lottery $p \in L(Z)$, let $G(p) = \sum_{z \in \text{good}} p(z)$ be the total probability that a prize in good occurs. $p \succsim q$ if and only if $G(p) \geq G(q)$.

- Let $Z = \{\$100, \$10, \$1\}$, good = $\{\$100\}$ and bad = $\{\$10, \$1\}$
- What is the preference over $p = (0.9, 0, 0.1)$ and $q = (0.5, 0.5, 0)$? He prefers p .

Example 3.3. Minimizing options

An individual wants the number of prizes that might be realized to be as small as possible. $p \succsim q$ if and only if $|\text{Supp}(p)| \leq |\text{Supp}(q)|$.

This preference relation makes sense for an individual who does not care about the realization of the lottery but wants to be as prepared as possible (physically or mentally) for all possible outcomes.

- Let $Z = \{\$100, \$10, \$1\}$
- What is the preference over $p = (0, 0, 1)$ and $q = (0.5, 0.5, 0)$? He prefers p .

Properties of Preferences

Continuity

Definition 3-2: Continuity

For any set Z of prizes, a preference relation $\succsim$ over $L(Z)$ is continuous if for any three prizes a, b and c in Z such that $[a] \succ [b] \succ [c]$ there is a number α with $0 < \alpha < 1$ such that $[b] \sim \alpha \cdot a \oplus (1-\alpha) \cdot c$.

- The intermediate value theorem: if $f: [0, 1] \rightarrow \mathbb{R}$ is continuous, then it takes on any given value between $f(0)$ and $f(1)$ at some point in $[0, 1]$.
- Suppose that for the prizes a, b and c we have $[a] \succ [b] \succ [c]$, and consider lotteries of the form $\alpha \cdot a \oplus (1-\alpha) \cdot c$ (with $0 \leq \alpha \leq 1$).
- The continuity property requires that as we move continuously from $\alpha=1$ (the degenerate lottery $[a]$) to $\alpha=0$ (the degenerate lottery $[c]$) we pass (at least once) some value of α such that the lottery $\alpha \cdot a \oplus (1-\alpha) \cdot c$ is indifferent to $[b]$.
- Small changes in probabilities do not change the nature of ordering between two lotteries.
- Let $Z = \{\$100, \$10, \$1\}$ and $\$100 \succ^* \$10 \succ^* \$1$. Does the periwink's preferences satisfy continuity? No.
- Let $Z = \{\$100, \$10, \$1\}$, good = $\{\$100\}$ and bad = $\{\$10, \$1\}$. Does the good and the bad preferences satisfy continuity? Yes.
- Let $Z = \{\$100, \$10, \$1\}$. Does the minimizing options preferences satisfy continuity? No.

Independence

- Let $Z = \{\$100, \$10, \$1\}$. What is the lottery q that gives $\$100$ and $\$1$ with even chances?
- $p = \frac{1}{2} \cdot [\$100] \oplus \frac{1}{2} \cdot [\$1] = (\frac{1}{2}, 0, \frac{1}{2})$.
- What is the (compound) lottery r that leads p and $q = [\$10]$ with even chances?
- $r = \frac{1}{2} \cdot p \oplus \frac{1}{2} \cdot q = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$

Definition 3-3: Compound lottery

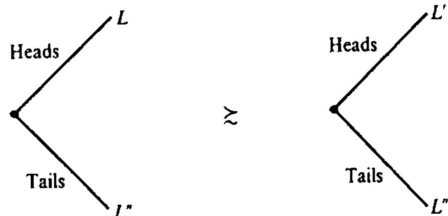
Let Z be a set of prizes, let $p_1, \dots, p_K$ be lotteries in $L(Z)$, and let $\alpha_1, \dots, \alpha_K$ be nonnegative numbers with sum 1. The **compound lottery** $\alpha_1 \cdot p_1 \oplus \dots \oplus \alpha_K \cdot p_K$ is the lottery that yields each prize $z \in Z$ with probability $\sum_K \alpha_K p_K(z)$.

Definition 3-4: Independence

Let Z be a set of prizes. A preference relation $\succsim$ over $L(Z)$ satisfies the **independence property** if for any lotteries $\alpha_1 \cdot z_1 \oplus \dots \oplus \alpha_K \cdot z_K$ and $\beta \cdot a \oplus (1-\beta) \cdot b$ we have

$$[z_K] \succsim \beta \cdot a \oplus (1-\beta) \cdot b \iff \alpha_1 \cdot z_1 \oplus \dots \oplus \alpha_K \cdot z_K \oplus \dots \oplus \alpha_K \cdot z_K \succsim \alpha_1 \cdot z_1 \oplus \dots \oplus \alpha_K \cdot (\beta \cdot a \oplus (1-\beta) \cdot b) \oplus \dots \oplus \alpha_K \cdot z_K$$

If we mix two lotteries with a third lottery, then the preference ordering of the two resulting mixtures does not depend on (is independent of) the particular third lottery used.



if and only if

$$L \succsim L'$$

- $\frac{1}{2} \cdot L \oplus \frac{1}{2} \cdot L''$ can be thought of as the compound lottery arising from a coin toss in which individual gets L if head comes up and L'' if tail does. Similarly, $\frac{1}{2} \cdot L' \oplus \frac{1}{2} \cdot L''$...
- Conditional on heads, $\frac{1}{2} \cdot L \oplus \frac{1}{2} \cdot L''$ is at least as good as $\frac{1}{2} \cdot L' \oplus \frac{1}{2} \cdot L''$. But conditional on tails, the two give identical results.
- Independence requires the "reasonable" conclusion that $\frac{1}{2} \cdot L \oplus \frac{1}{2} \cdot L''$ be at least as good as $\frac{1}{2} \cdot L' \oplus \frac{1}{2} \cdot L''$

Independence is at the heart of the theory of preferences under uncertainty. It exploits, in a fundamental manner, the structure of uncertainty present in the model. In the theory of consumer demand, there is no reason to believe that a consumer's preferences over various bundles of good 1 and 2 should be independent of the quantities of the other goods that she will consume. In the present context, however, it is natural to think that an individual's preferences between L and L' should determine which of the two she prefers to have as part of a compound lottery regardless of the other outcome, L'' . Why should L'' be irrelevant? She doesn't consume L or L' together with L'' but, rather, only instead of it.

Independence - Exercises

- Could you have one counterexample of independent preferences? Complementary goods
- Do the pessimistic preferences satisfy independence? No
- Do the "good and bad" preferences satisfy independence? Yes
- Do the "minimizing options" preferences satisfy independence? Yes

Monotonicity

Consider lotteries that assign positive probability to only two prizes a and b , with $[a] \succ [b]$. We say a preference relation over $L(Z)$ is monotonic if it ranks such lotteries by the probability a occurs. The next result says that any preference relation over $L(Z)$ that satisfies the independence property is monotonic.

Lemma 3-1: Independence implies monotonicity

Let Z be a set of prizes. Assume that $\succsim$, a preference relation over $L(Z)$, satisfies the independence property. Let a and b be two prizes with $[a] \succ [b]$, and let α and β two probabilities. Then: $\alpha > \beta \Leftrightarrow \alpha \cdot a \oplus (1-\alpha) \cdot b \succ \beta \cdot a \oplus (1-\beta) \cdot b$.

Expected Utility

Definition 3-5: Expected Utility

For any set Z of prizes, a preference relation $\succsim$ on $L(Z)$ is **consistent with expected utility** if there is a function $v: Z \rightarrow \mathbb{R}$ such that $\succsim$ is represented by the utility function U defined by: $U(p) = \sum_{z \in Z} p(z) v(z)$ for each $p \in L(Z)$. The function v is called the Bernoulli function for the representation.

EU (Expected Utility) is most commonly assumed in economic theory. How does EU individual compares lotteries? It attaches to each prize z a number, which we refer to as the value of the prize and denote $v(z)$. When evaluating a lottery p , it calculates the expected value of the lottery, $\sum_{z \in Z} p(z) v(z)$. The preferences are defined by: $p \succsim q$ if $\sum_{z \in Z} p(z) v(z) \geq \sum_{z \in Z} q(z) v(z)$.

Main Question: What is the relation between EU and preferences?

Proposition 3-1: Expected Utility is continuous and independent

A preference relation on a set of lotteries that is consistent with expected utility satisfies the continuity and independence properties.

Continuity and independence are **necessary** for a preference relation to have a EU representation.

Proof: Let Z be a set of prizes, let $\succsim$ be a preference relation over $L(Z)$, and let $v: Z \rightarrow \mathbb{R}$ be a function s.t. $U(p) = \sum_{z \in Z} p(z) v(z)$ represents $\succsim$.

- **Continuity**
- Let a, b and $c \in Z$ satisfy $[a] \succ [b] \succ [c]$.
- $U([z]) = v(z)$ for all $z \in Z$. Thus, $v(a) > v(b) > v(c)$.
- We can pick $\alpha \in [0, 1]$ s.t. $\alpha v(a) + (1-\alpha) v(c) = v(b)$.
- Then $U(\alpha \cdot a \oplus (1-\alpha) \cdot c) = U([b])$, i.e., $\alpha \cdot a \oplus (1-\alpha) \cdot c \sim [b]$.

$$EU \longrightarrow I + C$$

Exam

Independence

$$\begin{aligned} \alpha_1 \cdot z_1 \oplus \dots \oplus \alpha_k \cdot z_k \oplus \dots \oplus \alpha_K \cdot z_K &\succsim \alpha_1 \cdot z_1 \oplus \dots \oplus \alpha_k \cdot (\beta \cdot a \oplus (1-\beta) \cdot b) \oplus \dots \oplus \alpha_K \cdot z_K \\ &\iff \\ \alpha_1 V(z_1) + \dots + \alpha_k V(z_k) + \dots + \alpha_K V(z_K) &\succsim \alpha_1 V(z_1) + \dots + \alpha_k (\beta V(a) + (1-\beta) V(b)) + \dots + \alpha_K V(z_K) \\ &\iff \\ V(z_k) &\succsim \beta V(a) + (1-\beta) V(b) \\ &\iff \\ [z_k] &\succsim \beta \cdot a \oplus (1-\beta) \cdot b \end{aligned}$$

Exam

Expected Utility: Representation Theorem

Proposition 3-2: Continuity and Independence implies expected utility

A preference relation on a set of lotteries with a finite set of prizes that satisfies the continuity and independence properties is consistent with EU.

Continuity and independence are **sufficient** for a preference relation to have an EU representation.

Proof: Let Z be a finite set of prizes and let $\succsim$ be a preference relation on $L(Z)$ satisfying continuity and independence.

$I + C \rightarrow EU$

Label members of Z so that $[z_1] \succ \dots \succ [z_K]$. Let $z_1 = M$ and $z_K = m$.

First suppose $[M] \succ [m]$. Then, by continuity, for every prize z there is a number $V(z)$ s.t. $[z] \sim V(z) \cdot M \oplus (1-V(z)) \cdot m$.

By monotonicity, $V(z)$ is unique. Consider a lottery $P(z_1) \cdot z_1 \oplus \dots \oplus P(z_K) \cdot z_K$.

Applying independence K times, p is indifferent with $P(z_1) \cdot (V(z_1) \cdot M \oplus (1-V(z_1)) \cdot m) \oplus \dots \oplus P(z_K) \cdot (V(z_K) \cdot M \oplus (1-V(z_K)) \cdot m)$.

Above compound lottery is equal to: $(\sum_k P(z_k) V(z_k)) \cdot M \oplus (1 - \sum_k P(z_k) V(z_k)) \cdot m$.

Given $[M] \succ [m]$, by monotonicity, comparison between any two compound lotteries p and q is equivalent to the comparison between $\sum_k P(z_k) V(z_k)$ and $\sum_k Q(z_k) V(z_k)$. Now, suppose that $[M] \sim [m]$. Then, by independence, $p \sim [M]$ for any lottery p .

Choose $V(z_k) = 0$ for all k . $V(p) = 0$ represents the preference relation.

Exam

Expected Utility: Uniqueness

If $V: Z \rightarrow \mathbb{R}$ is the Bernoulli function for an expected utility representation of a certain preference relation over $L(Z)$.

Then, for any $\alpha > 0$ and β $W(z) = \alpha V(z) + \beta$ is a Bernoulli function for an expected utility representation.

Expected Utility: Discussion

EU is extremely convenient analytically (technical side). It is very easy to work with EU and very difficult without it.

It is pervasively used in Economics.

EU may provide a valuable guide to action (normative side): People often find it hard to think systematically about risky alternatives.

But if one believes that his choices should satisfy the properties on which the theorem is built (notably, the independence), then the theorem can be used as a guide in his decision process.

Theory and Experiments

There are 3 possible monetary prizes: 2.5 mln, 0.5 mln, 0.

Lottery $L_1 = (0, 1, 0)$ vs $L_1' = (0.1, 0.89, 0.01)$ $U(L_1) = 0.5 \text{ mln}$, $U(L_1') = 0.25 + 0.445 = 0.695 \text{ mln}$

Lottery $L_2 = (0, 0.11, 0.89)$ vs $L_2' = (0.10, 0, 0.9)$ $U(L_2) = 0.055 \text{ mln}$, $U(L_2') = 0.25 \text{ mln}$

It's common to express $L_1 \succ L_1'$ and $L_2' \succ L_2$. Is this behavior rational? Yes, even if it doesn't follow EU

$X = \{L_1, L_1', L_2, L_2'\}$. $L_1 \succ L_1' \succ L_2' \succ L_2$. Does this behavior satisfy EU? No, it violates independence.

$$L_1 \succ L_1' \implies 1U(\$1 \text{ mln}) > 0.89U(\$1 \text{ mln}) + 0.01U(\$0 \text{ mln}) + 0.1U(\$5 \text{ k})$$

$$L_2' \succ L_2 \implies 0.11U(\$1 \text{ mln}) + 0.89U(\$0 \text{ mln}) < 0.9U(\$0 \text{ mln}) + 0.1U(\$5 \text{ mln}) \rightarrow 0.11U(\$1 \text{ mln}) < 0.01U(\$0 \text{ mln}) + 0.1U(\$5 \text{ mln})$$

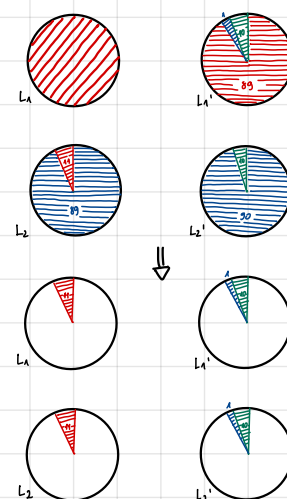
$$\rightarrow 1U(\$1 \text{ mln}) - 0.89U(\$1 \text{ mln}) < 0.01U(\$0 \text{ mln}) + 0.1U(\$5 \text{ mln}) \rightarrow 1U(\$1 \text{ mln}) < 0.89U(\$1 \text{ mln}) + 0.01U(\$0 \text{ mln}) + 0.1U(\$5 \text{ mln})$$

It violates the independence axiom, hence this preference relation can't have a EU representation.

This thought experiment is called the Allais Paradox:

① Choosing under uncertainty is a reflective activity in which one should be ready to correct mistakes if they are proven inconsistent with the basic principles embodied in the independence axiom.

② The Allais Paradox is of limited significance for economics as a whole because it involves probabilities close to 0 and 1.



③ Accommodates the paradox with a theory that defines preferences over larger and more complex objects than simply the ultimate lottery over outcome.

For example, regret theory.

④ Stick with the original choice domain of lotteries but to give up the independence axiom in favor of something weaker.

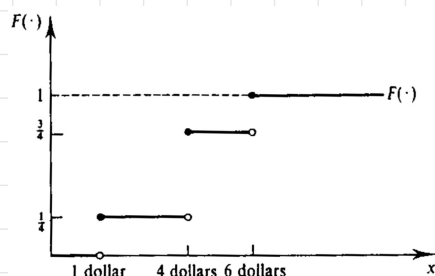
Money Lotteries and Risk Aversion

In many economic settings, individuals seem to display aversion to risk. We formalize risk aversion and study its properties. We assume that the set Z of prizes in the set of nonnegative real numbers, and think of the prize z as money $\$z$, a continuous variable.

The derivation of EU assumed a finite number of outcomes. However, the theory can be extended.

Denote amounts of money by the continuous variable z . A monetary lottery is a **cumulative distribution function** $F: \mathbb{R} \rightarrow [0, 1]$. $F(z)$ is the probability that the realized payoff is $\leq z$. If F admits a density function then $F(z) = \int_{-\infty}^z f(t) dt$.

The final distribution of a compound lottery $\alpha_1 \cdot F_1 \oplus \dots \oplus \alpha_K \cdot F_K$ is $F(z) = \sum_k \alpha_k F_k(z)$.



$$\begin{cases} \text{Prob}(1 \text{ dollar}) = \frac{1}{4} \\ \text{Prob}(4 \text{ dollars}) = \frac{1}{2} \\ \text{Prob}(6 \text{ dollars}) = \frac{1}{4} \end{cases} \rightarrow F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{4} & \text{if } x \in [1, 4) \\ \frac{3}{4} & \text{if } x \in [4, 6) \\ 1 & \text{if } x \geq 6 \end{cases}$$

Definition 3-6: Extending Expected Utility to Monetary Lotteries

$U(F) = \int v(z) dF(z)$. $U(\cdot)$ is the von-Neumann-Morgenstern (V.N-M) expected utility function defined on lotteries.

$v(\cdot)$ is the Bernoulli utility function defined on sure amounts of money.

Risk Aversion and its Measurement

The concept of risk aversion provides one of the central analytical techniques of economic analysis. We begin with a general definition that does not presume an EU formulation.

Definition 3-7: Risk Aversion and Risk Neutrality

If $Z = \mathbb{R}_+$, a preference relation $\succsim$ on $L(Z)$ is risk-averse if $[\int z dF(z)] \succsim F$ for every $F \in L(Z)$, is strictly risk-averse if the indifference holds only when the two lotteries are the same, is risk-neutral if $[\int z dF(z)] \sim F$ for every $F \in L(Z)$.

Risk aversion: for all lottery F , the corresponding degenerate lottery that yields the expected payoff with certainty is at least as good as the lottery F itself.

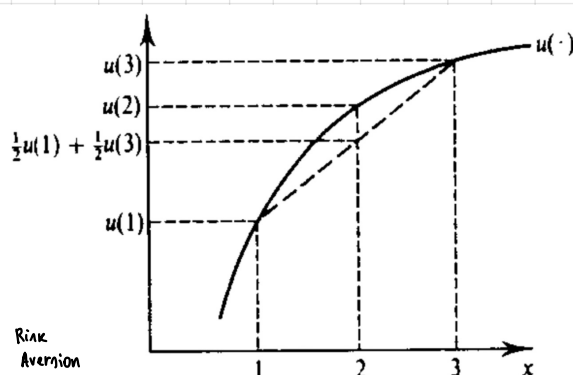
Now, let's add EU to the definition. If preferences admit an EU representation with Bernoulli function $v(\cdot)$, then it is risk-averse if and only if:

$\int v(z) dF(z) \leq v(\int z dF(z))$, Jensen's inequality. Defining property of a concave function. Under EU, risk aversion is equivalent to the concavity of $v(\cdot)$;

strict risk aversion is equivalent to the strict concavity of $v(\cdot)$.

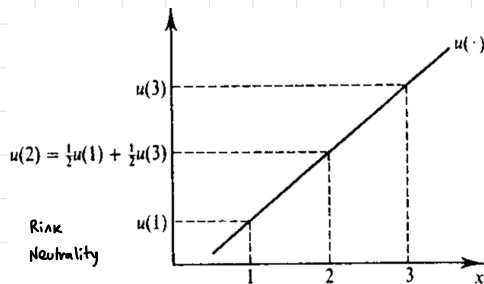
Strict concavity means marginal utility of money is decreasing. Utility gain from an extra dollar is smaller than utility loss of having a dollar less.

Therefore, a risk of gaining or losing a dollar with even probability is not worth taking.



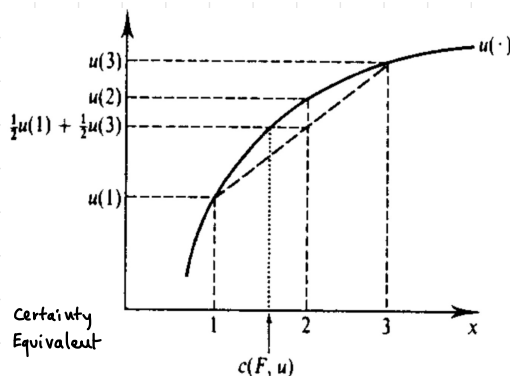
Under EU, risk neutral is equivalent to the linearity of $v(\cdot)$.

A useful concept for the analysis of risk aversion.



Definition 3-8: Certainty Equivalent

Given a Bernoulli function v , the certainty equivalent of F , denoted $c(F, v)$, is the amount of money for which the individual is indifferent between the lottery F and the certain amount of $c(F, v)$; that is, $v(c(F, v)) = \int v(z) dF(z)$



The figure shows geometric construction of $c(F, v)$ for an even probability lottery between 1 and 3.

Note that $c(F, v) < 2$, implying some expected return is traded for certainty.

$c(F, v) \leq \int z dF(z)$ for all F is equivalent to risk aversion.

$$c(F, v) \leq \int z dF(z) \Leftrightarrow u(c(F, v)) \leq u\left(\int z dF(z)\right) \Leftrightarrow \int u(z) dF(z) \leq u\left(\int z dF(z)\right)$$

Def. in class: given two V_1, V_2 . V_1 is more risk-averse than V_2 if: $c(F, V_1) \leq c(F, V_2)$ for all F .

given two V_1, V_2 V_1 is more risk-averse / more concave than V_2 if $\exists$ an increasing concave function ϕ , s.t. $V_1(z) = \phi(V_2(z))$

$V_2(z) = \ln z$, $\phi(x) = \ln x$, $\phi'(x) = \frac{1}{x}$, $\phi''(x) = -\frac{1}{x^2} < 0$ then $V_1(z) = \phi(V_2(z)) = \phi(\ln z) = \ln \ln z$.

Proposition 3-3: Equivalent properties of risk aversion

Suppose a preference relation $\succsim$ admits a EU representation with a Bernoulli utility function v on amounts of money. Then the following properties are equivalent:

- 1) $\succsim$ is risk-averse
- 2) v is concave.
- 3) $c(F, v) \leq \int z dF(z)$ for all F .

Once we know what it means to be risk-averse, we can try to measure the extent of risk-aversion.

Definition 3-9: Arrow-Pratt coefficient of absolute risk-aversion

Given a twice-differentiable Bernoulli utility function v for money, the Arrow-Pratt coefficient of absolute risk-aversion at z is defined as $r_A(z) = -v''(z) / v'(z)$.

Risk neutrality is equivalent to the linearity of v , that is, $v''(z) = 0$ for all z .

Risk aversion is equivalent to $v'' \leq 0$ when v is twice-differentiable.

Degree of risk aversion to be related to the curvature of v : risk aversion increases with the curvature of v at x , where $v''(z)$ is a measure of curvature of u at z .

But it is not invariant to positive linear transformation of v . To make it invariant, the simplest modification is to use $v''(z) / v'(z)$.

If we change sign to have a positive number for an increasing and concave v , we get: $-v''(z) / v'(z)$.

Now that we are equipped with a measure of risk aversion, we can compare risk attitudes. Given two Bernoulli utility function V_1 and V_2 , when can we say that V_2 is unambiguously more risk-averse than V_1 ?

Three plausible definitions: 1) $r_A(x, V_2) \geq r_A(x, V_1)$ for every x .

2) V_2 is more concave than V_1 . There exists an increasing concave function ϕ s.t. $V_2(z) = \phi(V_1(z))$.

3) $c(F, V_2) \leq c(F, V_1)$ for any F .

Proposition 3-4: Equivalent companions of risk aversion

Definitions (1), (2), (3) of the more-risk-averse-than relation are equivalent.

Companion Payoff Distributions

In the previous section, we compared utility functions. Now we compare payoff distributions.

What is the meaning of "the distribution F yields unambiguously higher return than the distribution G"?

Two sensible criteria suggest themselves: - First, whether **every** EV maximizer who values more over less prefers F to G.

- Second, for every amount of money z , the probability of getting at least z is higher under F than under G.

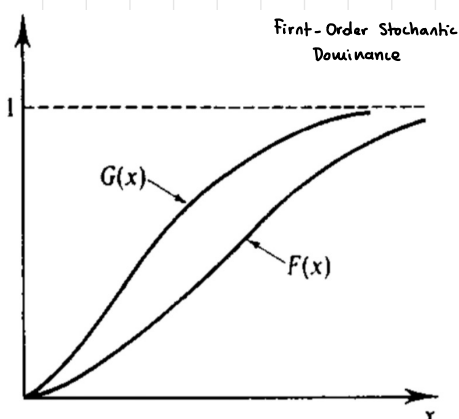
Fortunately, these two lead to the same concept.

Definition 3-10: First-Order Stochastic Dominance

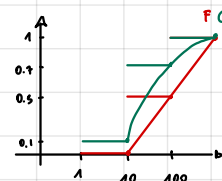
The distribution F first-order stochastically dominates G if, for every nondecreasing function $v: \mathbb{R} \rightarrow \mathbb{R}$, we have: $\int v(z) dF(z) \geq \int v(z) dG(z)$.

Proposition 3-5: Equivalent definitions

The distribution F first-order stochastically dominates G if and only if $F(z) \leq G(z)$ for every z .



- Let $z = \{\$1, \$10, \$100\}$. $F = (0, 0.5, 0.5)$ and $G = (0.1, 0.6, 0.3)$
- Does F FOSD G? Yes
- Is the FOSD relation complete? Transitive?



Subjective Expected Utility Theory (SEU)

Two different "probabilities": SuperEnalotto vs Horse race.

SuperEnalotto: Probabilities are observable, objective. Someone will pay 2€ to buy a SuperEnalotto. Probability of winning jackpot is 1 in 622,614,630.

Overall odds of winning a prize: 1 in 20.

Horse race: Probabilities are not observable, not objective. Say there are 3 horses in a race. Someone who places a 2€ bet on a horse A does not necessarily have a $1/3$ chance of winning.

We have been talking up to now about SuperEnalotto, but not horse race.

If we want to model situations of horse races, we cannot think about preferences over **lotteries**. Because we do not know probabilities. We need a different setup. We are going to think about **acts**. What is an act?

First, we need to define **states of the world**. We will do this with an example. Consider a race between three horses: A, B and C.

What are the possible outcome of this race? (Excluding ties).

State	Ordering
1	A, B, C
2	A, C, B
3	B, A, C
4	B, C, A
5	C, A, B
6	C, B, A

This is what we mean by states of the world. An exclusive and exhaustive list of all the possible realizations in a scenario.

An act is then an action which is defined by the outcome it gives in each state of the world.

Here are two examples:

- Act f: A 10€ even money bet that A will win.
- Act g: A 10€ bet at odds of 2 to 1 that C will win.

State	Ordering	Payoff Act f	Payoff Act g
1	A, B, C	\$10	-\$10
2	A, C, B	\$10	-\$10
3	B, A, C	-\$10	-\$10
4	B, C, A	-\$10	-\$10
5	C, A, B	-\$10	\$20
6	C, B, A	-\$10	\$20

So, how would you choose between acts f and g?

SEU answers the following. 1) Figure out the probability you would associate with each state of the world.

2) Figure out the utility you would gain from each prize.

3) Figure out the expected utility of each act according to those probabilities and utilities.

4) Choose the act with the highest utility.

So, in the above example. Utility from p : $[\pi(ABC) + \pi(ACB)]v(10) + [\pi(ABC) + \pi(BCA)]v(-10) + [\pi(CBA) + \pi(CAB)]v(-10)$

where π is the probability of each act.

Utility from g : $[\pi(ABC) + \pi(ACB)]v(-10) + [\pi(BAC) + \pi(BCA)]v(-10) + [\pi(CBA) + \pi(CAB)]v(20)$.

Suppose v is linear. Then p is preferred to g if: $\frac{[\pi(ABC) + \pi(ACB)]}{[\pi(CBA) + \pi(CAB)]} \geq \frac{3}{2}$ or the probability of A winning is more than $\frac{3}{2}$ times the probability of C winning.

Definition 3-11: SEU

Let Z be a set of prizes, Ω be a finite set of states of the world and F be the resulting set of acts (i.e., F is the set of all functions $f: \Omega \rightarrow Z$).

We say that preferences $\succsim$ on the set of acts F has a subjective expected utility representation if there exists a utility function $v: Z \rightarrow \mathbb{R}$ and probability function $\pi: \Omega \rightarrow [0,1]$ such that $\sum_{\omega \in \Omega} \pi(\omega) = 1$ and $f \succsim g \iff \sum_{\omega \in \Omega} \pi(\omega) v(f(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) v(g(\omega))$.

EU vs SEU

$$U(p) = \sum_{z \in Z} p(z) v(z) \quad \text{and} \quad U(f) = \sum_{\omega \in \Omega} \pi(\omega) v(f(\omega))$$

- Why would two DMs choose differently under EU? Because they might have different approaches to risk.
- Why would two DMs choose differently under SEU? Because they might have different Utility Functions due to different evaluation of the states of the world.

Consumer Preferences

In the first two topics, we discussed a model of preferences and rational choices over an arbitrary set of alternatives. (The most general model).

In the third topic, we apply the model to lotteries (Preferences under uncertainty and more structures, more results).

In the fourth topic, we apply the model to the choice of what to consume:

- A context central to standard economic theory. But, what is consumption?
- In economics, we mean by "consumption" a larger variety of activities that we mean by "consumption" in everyday life.
- When you eat an apple, you consume that apple, but also, when you wear a shirt, you consume that shirt.
- When we study rationality in consumption, we have in mind all these applications.

Is consumption a likely candidate for a successful application of the theory of rational choice? Do you give lots of thought to your choices about every day consumption, such as dinner purchases, or clothes purchases? Probably not... Indeed, some of them might be driven by addiction. But remember from the previous topic that by rationality we don't mean that people make carefully considered choices, we only mean that they act as if they maximized some consistent preferences. Perhaps, we don't think that our consumption is as if it maximized some well-defined preference, but to an outside observer it appears so.

Ultimately, data have to decide whether it is a good model.

Bundle of Goods

Definition 4-1-1: Set of alternatives (bundles)

The set of alternatives is $X = \mathbb{R}_+^2$. A member of X is a **bundle**.

- Why $\mathbb{R}_+^2$ and not $\mathbb{R}_+^n$? Most of the results are also true if there are a million goods rather than just two.
- Why $\mathbb{R}_+^2$ and not $\mathbb{N}_+^2$? 1/2 T-shirt? Modelling convenience.

Preferences over bundles

We now give some example of preference relations over bundles and represent them by indifference sets (or indifference curves).

The indifference set for the preference relation $\succsim$ and bundle a is $\{y \in X: y \sim a\}$, the set of all bundles indifferent to a .

The collection of all indifference sets is the partition induced by the equivalence relation $\sim$.

If $\succsim$ is represented by a utility function u , the indifference set for the bundle a is $\{y \in X: u(y) = u(a)\}$, the contour of u for the bundle a .

Here, "name utility" is analogous to "same elevation" in maps.

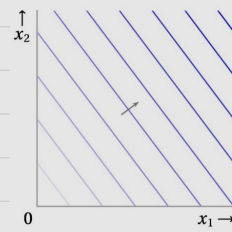
Example 4-1: Constant Tradeoff

The consumer has in mind two numbers v_1 and v_2 , where v_i is the value he assigns to a unit of good i .

His preference relation $\succsim$ is defined by the condition that $x \succsim y$ if $v_1 x_1 + v_2 x_2 \geq v_1 y_1 + v_2 y_2$.

Thus $\succsim$ is represented by the utility function $v_1 x_1 + v_2 x_2$. The indifference set for the bundle (a_1, a_2)

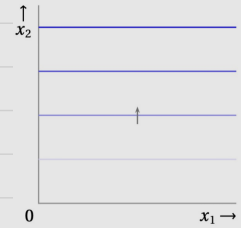
is $\{(x_1, x_2) : v_1 x_1 + v_2 x_2 = v_1 a_1 + v_2 a_2\}$, a line with slope $-v_1/v_2$.



Example 4-2: Only good 2 is valued

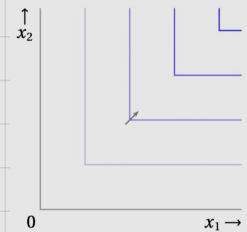
The consumer cares only about good 2, which he likes. His preference relation is represented by the utility function x_2 .

For this preference relation, every indifference set is an horizontal line.



Example 4-3: Complementary goods

The consumer wants the same amount of each good and prefers larger quantities. That is, he prefers a bundle x to a bundle y if and only if $\min\{x_1, x_2\} > \min\{y_1, y_2\}$. (Think of the goods as right and left shoes). Thus, $\min\{x_1, x_2\}$ is a utility function that represents this preference relation.



Example 4-4: Lexicographic preferences

The consumer cares primarily about the quantity of good 1, if this quantity is the same in two bundles, then he prefers the bundle with the larger quantity of good 2. Formally, $x \succsim y$ if either:

$$1) x_1 > y_1 \text{ or } 2) x_1 = y_1 \text{ and } x_2 \geq y_2.$$

For this preference relation, for any two bundles x and y we have $x \succ y$ or $y \succ x$, so that each indifference set consists of a single point.

Monotonicity

Definition 4-1-2: Monotone Preference Relation

The preference relation $\succsim$ on $\mathbb{R}_+^2$ is **monotone** if: $x_1 \geq y_1$ and $x_2 \geq y_2 \rightarrow (x_1, x_2) \succsim (y_1, y_2)$ and $x_1 > y_1$ and $x_2 \geq y_2 \rightarrow (x_1, x_2) \succ (y_1, y_2)$.

It is **strongly monotone** if: $x_1 \geq y_1$, $x_2 \geq y_2$ and $(x_1, x_2) \neq (y_1, y_2) \rightarrow (x_1, x_2) \succ (y_1, y_2)$.

Monotonicity is satisfied as long as commodities are "goods" rather than "bads".

If some is a bad, we may redefine it. For example, pollution is bad, we define consumption over "absence of pollution".

Continuity

Definition 4-1-3: Continuity

The preference relation $\succsim$ on $\mathbb{R}_+^2$ is **continuous** if whenever $x \succ y$ there exists a number $\varepsilon > 0$ such that for every bundle a for which the distance to x is less than ε and every bundle b for which the distance to y is less than ε we have $a \succ b$ (where the distance between any bundles (w_1, w_2) and (z_1, z_2) is $\sqrt{|w_1 - z_1|^2 + |w_2 - z_2|^2}$).

Continuous Preferences and Utility

Proposition 4-1-1: Continuous preference relation and continuous utility

A preference relation on $\mathbb{R}_+^2$ that can be represented by a continuous utility function is continuous.

Proof: Let $\succsim$ be a preference relation and let u be a continuous function that represents it.

Let $x \succ y$. Then $u(x) > u(y)$. $\varepsilon = \frac{1}{3}(u(x) - u(y))$.

By the continuity of u there $\exists \delta \in (0, \frac{1}{3}d(x, y))$ such that: $u(a) > u(x) - \varepsilon$ for all $a \in B_\delta(x)$

$u(b) < u(y) + \varepsilon$ for all $b \in B_\delta(y)$.

Thus $u(a) > u(x) - \varepsilon > u(y) + \varepsilon > u(b)$.

Proposition 4-1-2: Continuous preference relation and continuous utility.

Every continuous preference relation can be represented by a continuous utility function.

Convexity

Definition 4-1-4: Convex Preference Relation

The preference relation $\succsim$ on $\mathbb{R}_+^2$ is convex if: $a \succsim b \implies \lambda a + (1-\lambda)b \succsim b$ for all $\lambda \in (0,1)$.

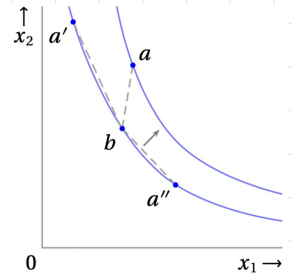
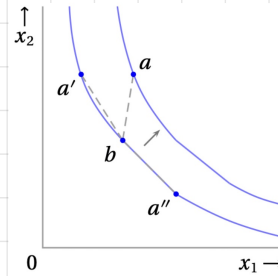
And strictly convex if $a \succ b$ and $a \neq b \implies \lambda a + (1-\lambda)b \succ b$ for all $\lambda \in (0,1)$.

Convexity: the trade-off that the consumer is willing to make among different goods.

A strong but central hypothesis in economics

A basic inclination for diversification:

- Under convexity, $x \sim y$, then $\frac{1}{2}x + \frac{1}{2}y \succsim x$.
- A realistic of economic life.
- "Economic theory would be in serious difficulty if this postulated propensity for diversification did not have significant descriptive content" - MWG.
- Could you think of a choice situation that violates convexity?



Example: Convexity on Lexicographic preferences

Lexicographic preferences are convex.

Assume $(a_1, a_2) \succ (b_1, b_2)$. If $a_1 > b_1$ then for every $\lambda \in (0,1)$ we have $\lambda a_1 + (1-\lambda)b_1 > b_1$ and thus $\lambda a + (1-\lambda)b \succ b$. If $a_1 = b_1$ then $\lambda a_1 + (1-\lambda)b_1 = b_1$. In this case $a_2 > b_2$ and hence $\lambda a_2 + (1-\lambda)b_2 > b_2$, so that $\lambda a + (1-\lambda)b \succ b$.

Convexity: Characterization 1

Proposition 4-1-3: Characterization of Convex preference relation

The preference relation $\succsim$ on $\mathbb{R}_+^2$ is convex if and only if for all $x^* \in X$ the upper contour set of x^* $\{x \in X : x \succsim x^*\}$ (containing all bundles at least as good as x^*) is convex.

Proof: Only if. Assume that $\succsim$ is convex. Let $a, b \in \{x \in X : x \succsim x^*\}$. Without loss assume $a \succ b$. Then for $\lambda \in (0,1)$, $\lambda a + (1-\lambda)b \succ b \succsim x^*$.
 If. Assume that the set $\{x \in X : x \succsim x^*\}$ is convex for all x^* . If $a \succ b$ then we have $a \in \{x \in X : x \succsim b\}$.
 $b \in \{x \in X : x \succsim b\}$. Convexity gives $\lambda a + (1-\lambda)b$ in the set. Thus $\lambda a + (1-\lambda)b \succ b$.

Exam

Mathematically, convexity is equivalent to convex indifference curve. What does convexity mean in economic terms?

It means that the amount of good 2 that the consumer is willing to give up for one more unit of good 1 is getting smaller as the consumer has more and more of good 1. Very loosely speaking, the marginal utility of good 1, expressed in units of good 2, is diminishing: the more of good 1 the consumer has, the less is an additional unit of good 1 worth to her.

Convexity: Decreasing Marginal Rate of Substitution

Consider three bundles $a = (10, 10)$, $b = (11, 10 - \beta)$, and $c = (12, 10 - \beta - \gamma)$ for which $a \sim b \sim c$.

When the amount of good 1 increases from 10 to 11, the consumer is kept indifferent by reducing the amount of good 2 by β .

When the amount of good 1 increases from 11 to 12, the consumer is kept indifferent by reducing the amount of good 2 by γ .

Strong monotonicity and convexity implies decreasing marginal rate of substitution, $\beta \geq \gamma$.

The rate at which good 2 is substituted for good 1 decreases as the amount of good 1 increases.

Suppose, to the contrary that $\beta < \gamma$. Then $\beta < \frac{1}{2}(\beta + \gamma) \implies (11, 10 - \frac{1}{2}(\beta + \gamma)) \prec b = (11, 10 - \beta)$. But $(11, 10 - \frac{1}{2}(\beta + \gamma)) = \frac{1}{2}a + \frac{1}{2}c \succsim c \sim b$.

Convexity: Characterization 2

Proposition 4-1-4: Preferences with concave representation are convex

A preference relation $\succsim$ on $\mathbb{R}_+^2$ that is represented by a concave function is convex.

Proof: Let $\succsim$ be a preference relation represented by a concave function u .

Assume that $a \succsim b$, so that $u(a) \geq u(b)$.

By the concavity of u , $u(\lambda a + (1-\lambda)b) \geq \lambda u(a) + (1-\lambda)u(b) \geq u(b)$.

Thus, $\lambda a + (1-\lambda)b \succsim b$.

Exam

Convexity of $\succsim$ does not imply u is concave. There may not be any concave utility function representing a particular convex preference relation $\succsim$.

Definition 4-1-5: Quasiconcave function

$u: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is quasiconcave if $u(\alpha a + (1-\alpha)b) \geq \min\{u(a), u(b)\}$ for any $x, y \in \mathbb{R}_+^2$ and $\alpha \in [0, 1]$.

Proposition 4-1-5: Convex preferences and Quasiconcave utility

Suppose the preference relation $\succsim$ on $\mathbb{R}_+^2$ can be represented by utility function u . $\succsim$ is convex if and only if u is quasiconcave.

Consumer Choice

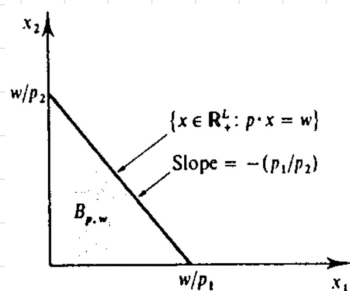
Consumption set

- Consumption choices are limited to those bundles one can afford.
- The affordability of a bundle depends on two things: market prices p and wealth w .
- Goods are traded in the market at dollar prices that are publicly quoted. $p = (p_1, p_2)^T \in \mathbb{R}_+^2$.
 - Dollar cost of a unit of each good.
 - A negative price means a buy is paid to consume.
 - For simplicity, we assume $p \gg 0$.
- **Price-taking Assumption:** Prices are beyond the influence of the consumer.
- $x \in X$ is affordable if: $p \cdot x = p_1 x_1 + p_2 x_2 \leq w$.

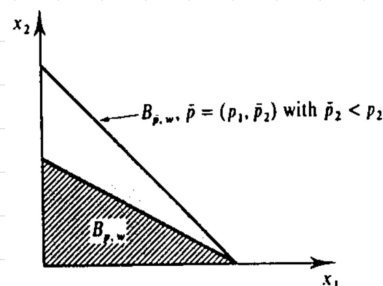
Definition 4-2-0: Walrasian budget set

The Walrasian, or competitive budget set $B_{p,w} = \{x \in X : p \cdot x \leq w\}$ is the set of all feasible bundles for the consumer who faces market prices p and has wealth w .

Given p and w , the consumer's choice problem is to choose a bundle x from $B_{p,w}$.



- Budget Hyperplane: $\{x \in X : p \cdot x = w\}$
- $B_{p,w}$ is convex.



Demand Functions and Comparative Statistics

- A Walrasian demand function is a function x that assigns to each budget set one of its members.
- Define $x((p_1, p_2), w)$ to be the bundle assigned to the budget set $B((p_1, p_2), w)$.

The definition does not assume that the demand function is the result of the consumer's maximizing a preference relation. In the next page we propose some examples.

- 1) The consumer purchases only the cheaper good; if the prices of the goods are the same, he divides his wealth equally between the two.
- 2) The consumer chooses the same quantity of each good and spends all his wealth.
- 3) The consumer spends half of his wealth on each good.
- 4) The consumer buys as much as he can of good 1 up to 7 units and with any wealth remaining buys good 2.

$$1) x(p, w) = \begin{cases} (w/p_1, 0) & \text{if } p_1 < p_2 \\ (w/2p_1, w/2p_2) & \text{if } p_1 = p_2 \\ (0, w/p_2) & \text{if } p_1 > p_2 \end{cases}$$

$$2) = 3) x(p, w) = \left(\frac{w}{2p_1}, \frac{w}{2p_2} \right)$$

$$4) x(p, w) = \begin{cases} (w/p_2, 0) & \text{if } w \leq 7p_1 \\ (7, \frac{w-7p_1}{p_2}) & \text{if } w > 7p_1 \end{cases}$$

We now introduce some desirable properties demand functions may satisfy.

Definition 4-2-1: Homogeneous of degree zero

The Walrasian demand function $x(p, w)$ is homogeneous of degree zero if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

- Choice depends only on the net of feasible points.
- Can you think of any example of violations? Yes, when you're earning in one country and spending in a poorer/richer one.

Definition 4-2-2: Walras' Law

The Walrasian demand function $x(p, w)$ satisfies Walras' Law if, for every $p \gg 0$ and $w > 0$, we have $p \cdot x = w \quad \forall x \in x(p, w)$.

- The consumer fully expends his wealth.
- Don't we have savings so that budget is intertemporal?
- Fully expends wealth over lifetime.

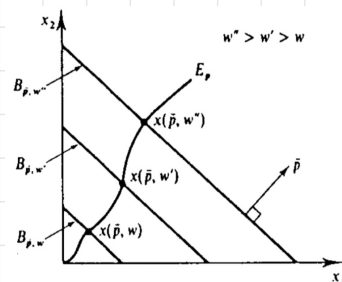
$$x_1(p, w) = \frac{p_2}{p_1 + p_2} \cdot \frac{w}{p_1}, \quad x_2(p, w) = \frac{p_1}{p_1 + p_2} \cdot \frac{w}{p_2}$$

- Does $x(p, w)$ satisfy homogeneity of degree zero when $\beta = 1$? $x(p, w) = x(\lambda p, \lambda w) \rightarrow x_1(\lambda p, \lambda w) = \frac{\lambda p_2}{\lambda p_1 + \lambda p_2} \cdot \frac{\lambda w}{\lambda p_1} = x_1(p, w)$. So, Yes.
- Does $x(p, w)$ satisfy Walras' Law when $\beta = 1$? $p_1 \cdot x_1 + p_2 \cdot x_2 = w \rightarrow p_1 \cdot \frac{p_2}{p_1 + p_2} \cdot \frac{w}{p_1} + p_2 \cdot \frac{p_1}{p_1 + p_2} \cdot \frac{w}{p_2} = \frac{p_2 + p_1}{p_1 + p_2} w = w$. So, Yes.
- How about the case $\beta \in (0, 1)$?

Wealth Effect

Comparative statics: how choice varies with change in wealth and prices.

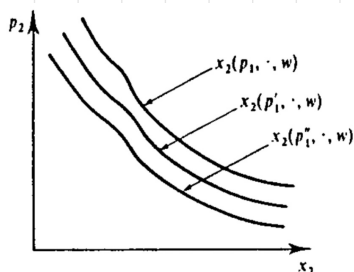
- **Wealth effect** for good i : at any (p, w) , $\partial x_i(p, w) / \partial w$
- A good is **normal** at (p, w) if $\partial x_i(p, w) / \partial w \geq 0$.
- A good is **inferior** at (p, w) if $\partial x_i(p, w) / \partial w < 0$.
- If both goods are normal at all (p, w) , then **demand is normal**.
- When goods are large aggregates (food, shelter), instead of disaggregated (one kind of shoes).
- $D_w x(p, w) = (\partial x_1(p, w) / \partial w, \partial x_2(p, w) / \partial w)^T$



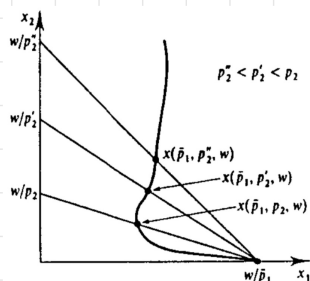
Price Effect

- How consumption change as prices vary?
- Suppose w and p_1 are fixed

$$x_2(p_2, \cdot, w)$$



Offer Curve: locus of points demanded in X as we range over all possible p_2 .



- Price effect of P_i on good j : $\partial x_j(P, w) / \partial P_i$.
- $\partial x_i(P, w) / \partial P_i \leq 0$? Not necessary
- Good i is said to be a **Giffen good** at (P, w) if $\partial x_i(P, w) / \partial P_i > 0$.

Giffen goods

Low-quality goods may be Giffen for consumers with low wealth.

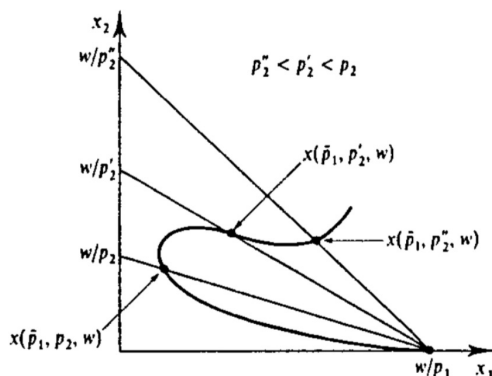
Imagine a poor consumer initially is having potatoes because they are a low-cost way to avoid hunger.

If price of potatoes fall, he can afford to buy other, more desirable foods that also keep him from being hungry.

His consumption of potatoes may well fall as a result.

This story involves a wealth consideration: When price of potato falls, the consumer is effectively wealthier, he can afford to purchase more generally.

How about law of demand? It defy standard economic and consumer demand theory determining a downward sloping demand curve.



Is food 2 a Giffen good at $(\bar{P}_1, P_2', w)$?

Weak Axiom of Revealed Preference

Now, we bring rationality back to the picture.

Definition 4-2-3. Weak Axiom of Revealed Preference (WA)

The Walrasian demand function $x(P, w)$ satisfies the weak axiom of revealed preference if the following property holds for any (P, w) and (P', w') .

If $P \cdot x(P', w') \leq w$ and $x(P', w') \neq x(P, w)$, then $P' \cdot x(P, w) > w'$.

Idea behind:

- $P \cdot x(P', w') \leq w$ and $x(P', w') \neq x(P, w)$ means that when facing (P, w) , consumer chose $x(P, w)$ even though $x(P', w')$ was also affordable.
- This choice "reveals" a preference for $x(P, w)$ over $x(P', w')$.
- Suppose we expect the consumer to display some consistency in his demand behavior.
- He would choose $x(P, w)$ over $x(P', w')$ whenever they are both affordable.
- If no, bundle $x(P, w)$ must NOT be affordable at the price-wealth combination (P', w') at which the consumer chooses $x(P', w')$.
- That is, we must have $P' \cdot x(P, w) > w'$.

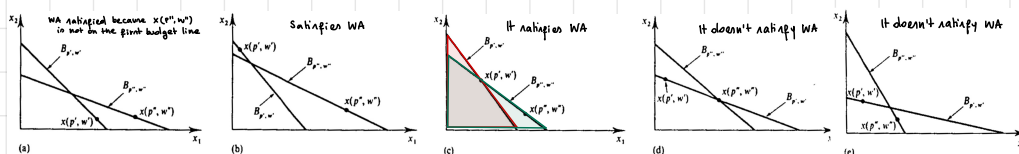
Proposition 4-2-1: Demand function of rational consumer satisfies WARP

A Walrasian demand function $x(P, w)$ that is rationalized by a monotone preference relation $\succeq$ satisfies the weak axiom of revealed preference.

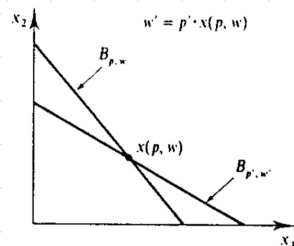
Proof:

Let x be a demand function that is rationalized by the monotone preference relation $\succeq$. Assume, contrary to the result, that (i) a is revealed to be at least as good as b and (ii) b is revealed to be better than a . Given (i) we have $a \succeq b$. By (ii) there are prices P_1 and P_2 and wealth w such that $b = x(P_1, P_2, w)$ and $P_1 a_1 + P_2 a_2 < w$. Let c be a bundle in $B((P_1, P_2), w)$ with $c_1 > a_1$ and $c_2 > a_2$. By the monotonicity of the preference relation we have $c \succ a$, and since b is chosen from $B((P_1, P_2), w)$ we have $b \succeq c$. It follows from the transitivity of the preference relation that $b \succ a$, contradicting $a \succeq b$.

Examples:



- Implication:**
- WA has significant implications for the effect of price changes on demand.
 - We need to concentrate, however, on a special kind of price change.
 - Price changes affect consumer in two ways:
 - Alter the relative cost of different goods.
 - Also change real wealth.
 - We isolate the first effect.
 - Imagine a situation in which change in prices is accompanied by a change in wealth that initial consumption bundle just affordable at new prices.
 - (P, w) and $x(P, w)$ initially. $P \rightarrow P'$. Let $w' = P' \cdot x(P, w)$.
 - $\Delta W = \Delta P \cdot x(P, w)$ where $\Delta P = (P' - P)$. **Slutsky Wealth Compensation.**



Proposition 4-2-2: WA and Compensated price changes

Suppose that $x(P, w)$ is homogeneous of degree zero and satisfies Walras' law. Then $x(P, w)$ satisfies the weak axiom if and only if the following property holds: For any compensated price change from an initial situation (P, w) to a new price-wealth pair $(P', w') = (P', P' \cdot x(P, w))$, we have:

$$(P' - P) \cdot [x(P', w') - x(P, w)] \leq 0 \quad \text{with strict inequality whenever } x(P', w') \neq x(P, w). \quad (1)$$

Proof: Only if Part: (WA $\rightarrow$ Compensated Law of Demand)

- Suppose $x(P', w') \neq x(P, w)$
- The LHS of (1) can be written as: $\underbrace{P' \cdot [x(P', w') - x(P, w)]}_{= 0} - P \cdot [x(P', w') - x(P, w)] \quad (2)$
- Walras' law, $w' = P' \cdot x(P', w')$.
- Compensated price change, $P' \cdot x(P, w) = w'$.
- First term is zero.
- $x(P, w)$ is affordable under (P', w') .
- WA implies $x(P', w')$ must NOT be affordable under (P, w) : $P \cdot x(P', w') > w$.
- Second term > 0 .

Exam

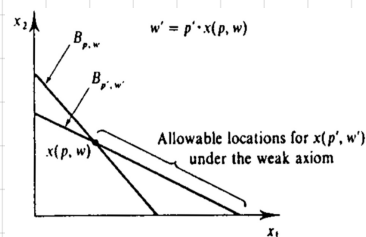
Compensated Law of Demand

Inequality (1) can be written as $\Delta P \cdot \Delta x \leq 0$ where $\Delta P = (P' - P)$ and $\Delta x = [x(P', w') - x(P, w)]$.

Demand and price move in opposite directions. Law of demand holds for compensated price changes

Compensated Law of Demand: Consider the simple case $\Delta P = (\Delta P_1 < 0, 0)$

- $\Delta P \cdot \Delta x = \Delta P_1 \Delta x_1 < 0$.
- $\Delta x_1 > 0$.



- Law of Demand? WA is not sufficient
- The consistency requirement embodied in the weak axiom (combined with homogeneity of degree zero and Walras' law) is equivalent to the compensated law of demand, not "uncompensated" law of demand.

Testing Rationality in practice

- WA/Rationality is a black and white criterion. It determines either that a consumer is rational, or that she is not. There is no grey area.
- Another question that we could ask is whether these violations are "serious". But what is "serious"?
- This is a good question, but one that we have to leave open here.
- How do people choose their consumption in practice?
- Beatty, T.K., & Crawford, I.A. (2011). How demanding is the revealed preference approach to demand?

Their work studies the consumption behavior of Spanish households between 1985 and 1997, using a government collected expenditure survey. They have observations on 3,134 households. The data report expenditures not for individual goods, but for "aggregates", such as "food".

The first result that Beatty and Crawford report is that 95.7% of the observed households pass the rationality test. That is an extraordinary large number.

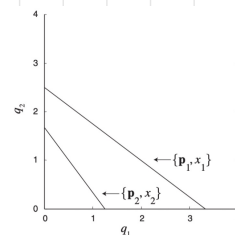
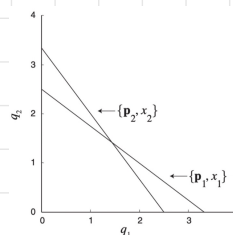
The result matches observations in other studies. Previous research has found predominantly rational behavior among New York dairy farmers, Danish consumers,

children and capuchin monkeys. The paper goes further, though. It asks, why it seems that everyone in the world is rational.

They compare the rational proportion among real the Spanish households with that of a similar, fictional group of households, who have the same budget sets as the Spanish households, but who randomly pick their consumption. This group of households can just be simulated on the computer. They discover that those households would also pass the rationality test with high rate, only 4.5% lower than the observed pass rate of the Spanish households.

Caution: don't interpret the high number of people passing the rationality test as a "triumph of economic theory"

Why do they have the finding that even random households behave rationally?



Most of the time budget sets are nested, as the right figure. That is, one budget set is just larger than the other. Price fluctuations in their set of observations are small, but income variation are large. If one budget set is larger than another, whatever choice the household makes satisfies the axioms of revealed preferences.

What does all of this teach us?

- 1) Approaching data through the lenses of the theory that we have studied is likely to be successful.
- 2) However, that does not tell us much about the "true" rationality of people.

But it is useful to rely on the economic model of consumer behavior when studying consumer choices in practice? Or should we perhaps adopt a random choice model? One advantage of the economic approach is that it allows us to attribute preferences to people.

Those preferences then allow us to assess how well the market satisfies those preferences. Having an answer to this question is extremely useful when thinking about public policy and the market. Economists proceed in this way all the time. Are the preferences that we attribute to consumers really those that they reveal to us through their choices. But what else do we have to work with?

Consumer Demand

Utility Maximization Problem Set

We now turn to the study of the consumer's decision problem under the assumptions that:

- the consumer has a continuous and monotone preference relation.
- $u(x)$ is a continuous utility representing these preferences.

Definition 4-3-1: Utility Maximization Problem

The consumer's problem of choosing her most preferred bundle given prices $p \gg 0$ and wealth $w > 0$ can now be stated as the following

Utility Maximization Problem (UMP): $\max_{x \geq 0} u(x)$ s.t. $p \cdot x \leq w$.

Exercise $\max_{x \geq 0} u(x) = x_1 x_2$ s.t. $2x_1 + x_2 \leq 4$.

$$2x_1 + x_2 = 4 \rightarrow x_2 = 4 - 2x_1 \rightarrow u(x) = x_1(4 - 2x_1) = -2x_1^2 + 4x_1 \rightarrow \partial u / \partial x_1 = -4x_1 + 4 = 0 \begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases} \text{ then } \begin{cases} x^* = (1, 2) \\ u(x^*) = 1 \cdot 2 = 2 \end{cases}$$

Remarks: 1) Indeed there is a solution. A general feature of UMP.

2) A solution gives us the optimal consumption bundles and the maximal utility value.

Proposition 4-3-1: If $p \gg 0$ and $u(\cdot)$ is continuous, then UMP has a solution.

Proof: If $p \gg 0$, then the budget set $B_{p,w} = \{x \in \mathbb{R}_+^2 : p \cdot x \leq w\}$ is a bounded and closed set.

Any bounded and closed set in $\mathbb{R}^2$ is compact.

A continuous function has a maximum value on any compact set.

Exam

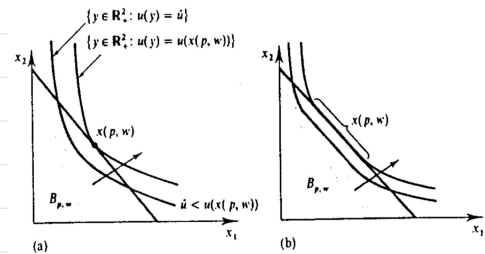
Two objects emerge from UMP:

- 1) The consumer's set of optimal consumption bundles, the solution of UMP.
- 2) The consumer's maximal utility value, the value function of UMP.

We now focus our attention on the properties of these two.

Walrasian Demand Function

- Walrasian Demand correspondence:** The rule that assigns the set of optimal bundle in UMP to each $(p, w) \gg 0$ is denoted by $x(p, w) \in \mathbb{R}_+^2$.
- Walrasian Demand function:** Singled valued $x(p, w)$



Proposition 4-3-2: Suppose that $u(\cdot)$ is a continuous utility function representing a monotone preference relation $\succsim$ defined on $\mathbb{R}_+^2$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:

- Homogeneity of degree zero** in (p, w) : $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar $\alpha > 0$.
- Walras' Law:** $p \cdot x = w$ for all $x \in x(p, w)$.
- Convexity/Uniqueness:** If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Proof Homogeneity: $B_{\alpha p, \alpha w} = \{x \in \mathbb{R}_+^2 : \alpha p \cdot x \leq \alpha w\} = \{x \in \mathbb{R}_+^2 : p \cdot x \leq w\} = B_{p, w}$

Note that this property does not require any assumption on u .

Homogeneity was an assumption in Topic 4-2, but a property here.

Exam

Proof Walras' Law: Proof by contradiction.

Suppose $p \cdot x < w$ for some $x \in x(p, w)$. Then there must exist another bundle $y = (x_1 + \varepsilon, x_2 + \varepsilon)$ where $\varepsilon > 0$ is small enough such that $p \cdot y < w$. By monotonicity of $\succsim$, $y \succ x$. This contradicts x being optimal in UMP.

Exam

Proof Convexity/Uniqueness (Part I): Suppose that u is quasiconcave and that there are two bundles $x, x' \in x(p, w)$ and $x \neq x'$.

Denote $u(x) = u(x') = u^*$. WNTS $x'' = \alpha x + (1-\alpha)x' \in x(p, w)$ for any $\alpha \in [0, 1]$.

By quasiconcavity, $u(x'') \geq \min\{u(x), u(x')\} = u^*$.

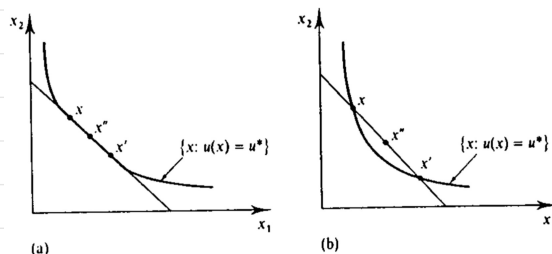
$p \cdot x'' = p \cdot [\alpha x + (1-\alpha)x'] \leq w$. x'' is feasible in UMP. Hence, $x'' \in x(p, w)$.

Proof Convexity/Uniqueness (Part II): Proof by contradiction.

Suppose that u is strictly quasiconcave and that there are two bundles $x, x' \in x(p, w)$ and $x \neq x'$. Denote $u(x) = u(x') = u^*$.

By strict quasiconcavity, $u(x'') > \min\{u(x), u(x')\} = u^*$. $p \cdot x'' = p \cdot [\alpha x + (1-\alpha)x'] \leq w$. x'' is feasible in UMP.

This contradicts with the assumption x, x' are optimal. We conclude there can be at most one element in $x(p, w)$.



Example: UMP with Cobb-Douglas Utility function. $\max_{x \geq 0} x_1^\alpha x_2^{1-\alpha}$ s.t. $p_1 x_1 + p_2 x_2 \leq w$.

$$\max_{x \geq 0} x_1^\alpha x_2^{1-\alpha} \text{ s.t. } p_1 x_1 + p_2 x_2 = w \rightarrow \max_{x \geq 0} x_1^\alpha (w - p_1 x_1)^{1-\alpha} \rightarrow \text{FOC: } \alpha x_1^{\alpha-1} (w - p_1 x_1)^{1-\alpha} + x_1^\alpha (1-\alpha) (w - p_1 x_1)^{-\alpha} (-p_1) = 0$$

$$\rightarrow \alpha x_1^{\alpha-1} (w - p_1 x_1)^{1-\alpha} = p_1 x_1^\alpha (1-\alpha) (w - p_1 x_1)^{-\alpha}$$

$$\rightarrow \alpha (w - p_1 x_1) = (1-\alpha) p_1 x_1 \rightarrow x_1 = \frac{\alpha w}{p_1} > 0 \text{ and } x_2 = \frac{(1-\alpha) w}{p_2}$$

The expenditure on each commodity is a constant factor of wealth for any price vector p .

Exercise: Verify the three properties for the Walrasian Demand function generated by the Cobb-Douglas utility functions.

Definition: Marginal Rate of Substitution. Suppose that $u(\cdot)$ is a differentiable utility function representing a monotone preference relation $\succsim$ defined by $\mathbb{R}^2_+$. The marginal rate of substitution at bundle x , denoted $MRS(x)$ is $u_1(x)/u_2(x)$ where $u_i(x)$ is the marginal utility w.r.t. good i .

- MRS is also the slope of an indifference curve (more precisely, to the slope multiplied by -1) passing through x .
- $u(x) = c$, $u_1(x)dx_1 + u_2(x)dx_2 = 0$, $-\frac{dx_2}{dx_1} = \frac{u_1(x)}{u_2(x)}$

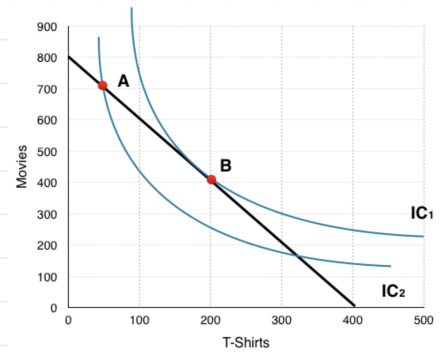
Proposition 4-3-3: Marginal rate of substitution and price ratio. Suppose that $u(\cdot)$ is a differentiable utility function representing a monotone preference relation $\succsim$ defined by $\mathbb{R}^2_+$. If $x^* \in x(p, w)$, then:

- ① $x_1^* > 0$ and $x_2^* > 0 \Rightarrow MRS(x^*) = p_1/p_2$
- ② $x_1^* = 0 \Rightarrow MRS(x^*) \leq p_1/p_2$, i.e., $x_1^* = 0 \wedge x_2^* > 0 \Rightarrow \frac{MU_1}{p_1} \leq \frac{MU_2}{p_2}$
- ③ $x_2^* = 0 \Rightarrow MRS(x^*) \geq p_1/p_2$, i.e., $x_1^* > 0 \wedge x_2^* = 0 \Rightarrow \frac{MU_1}{p_1} \geq \frac{MU_2}{p_2}$

Proof (Part 1):

- Suppose that $x_1^* > 0$, $x_2^* > 0$, and $MRS(x^*) < p_1/p_2$.
- Let $x' = x^* + (-\varepsilon, \varepsilon p_1/p_2)$ for small enough $\varepsilon > 0$.
- x' is on the budget line, i.e., $p \cdot x' = w$.
- $u(x') = u(x^*) + (x_1' - x_1^*)u_1(x^*) + (x_2' - x_2^*)u_2(x^*) + o^2$.
- $u(x') - u(x^*) = -\varepsilon u_1(x^*) + \varepsilon p_1/p_2 u_2(x^*) + o^2 > 0 + o^2$
- Thus, x' is feasible and achieves higher utility level than x^* , contradicting the fact that x^* is a solution of UMP.

Exam



Proof (Part 2, 3)

- Similar arguments establish part 2 and 3.
- We forego the proofs (covered in the exam).

Exam

Example: UMP with Cobb-Douglas utility function. $\max_{x \geq 0} x_1^\alpha x_2^{1-\alpha}$ s.t. $p_1 x_1 + p_2 x_2 \leq w$.

Exercise: Solve it using the relationship between MRS and price ratio.

Exercise: UMP with linear utility function $\max_{x \geq 0} x_1 + x_2$ s.t. $p_1 x_1 + p_2 x_2 \leq w$.

Indirect Utility Function

u = utility
 v = indirect utility

Indirect Utility Function: for each $(p, w) \gg 0$, the utility level of the UMP is denoted $v(p, w) \in \mathbb{R}$. It is equal to $u(x^*)$ for any $x^* \in x(p, w)$.

Proposition 4-3-4: Suppose that $u(\cdot)$ is a continuous utility function representing a monotone preference relation $\succsim$ defined on $\mathbb{R}^2_+$. Then, the indirect utility function is:

- ① Homogeneity of degree zero in (p, w) . $v(\alpha p, \alpha w) = v(p, w)$ for any (p, w) and scalar $\alpha > 0$.
- ② Strictly increasing in w and nonincreasing in p_1 and p_2 .
- ③ Quasiconvex: the set $\{ (p, w) : v(p, w) \leq \bar{v} \}$ is convex for any $\bar{v}$.
- ④ Continuous in p and w .

- Property 3 says v is quasiconvex, not quasiconcave. And it does not require u to be quasiconcave.

Proof: Property 1 and 2 follow readily from previous discussion.

- We forego the proof of 4 (not covered in exam)
- To prove 3, $v(p, w)$ is quasiconvex, suppose that $v(p, w) \leq \bar{v}$ and $v(p', w') \leq \bar{v}$.
- For any $\alpha \in [0, 1]$, consider $(p'', w'') = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$.
- WNTS $v(p'', w'') \leq \bar{v}$, i.e., for any x with $p'' \cdot x \leq w''$, we have $u(x) \leq \bar{v}$.
- $p'' \cdot x \leq w'' \Rightarrow \alpha p \cdot x + (1-\alpha)p' \cdot x \leq \alpha w + (1-\alpha)w'$.
- Either $p \cdot x \leq w$ or $p' \cdot x \leq w'$ (or both).
- If $p \cdot x \leq w$, then $u(x) \leq v(p, w) \leq \bar{v}$
- If $p' \cdot x \leq w'$, then $u(x) \leq v(p', w') \leq \bar{v}$

Exam

The Walrasian demand function does not depend on the utility representation.

The indirect utility function does: • If $v(p, w)$ is the indirect utility when utility function is u .

• Then the indirect utility function corresponding to utility representation $\bar{u}(x) = f(u(x))$ is $\bar{v}(p, w) = f(v(p, w))$.

Expenditure Minimization Problem

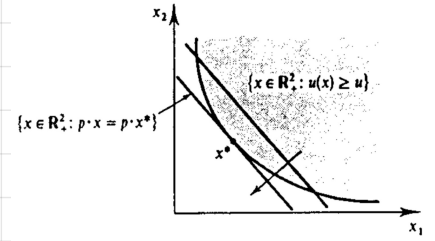
In this section, we study the following **expenditure minimization problem (EMP)**: $\min_{x \geq 0} p \cdot x$ s.t. $u(x) \geq u$

VMP: the maximal level of utility that can be obtained given wealth level.

EMP: the minimal level of wealth required to reach utility level u .

The naive aim of efficient use of purchasing power while reversing the roles of objective function and constraint.

Throughout this section, we assume u is continuous and monotone.



x^* : the least costly bundle that still allow the consumer to achieve the utility level u . It lies on the lowest possible budget line in the feasible set.

Exercise: $\min_{x \geq 0} 2x_1 + x_2$ s.t. $u(x) = x_1 x_2 \geq 2$. Solution: $x^* = (1, 2)$, $p \cdot x^* = 4$

$\max_{x \geq 0} u(x) = x_1 x_2$ s.t. $2x_1 + x_2 \leq 4$. Solution: $x^* = (1, 2)$, $u(x^*) = 2$

VMP and EMP are closely related.

Relationship between EMP and VMP

Proposition 4-3-5 Suppose that u is a continuous utility function representing a monotone preference relation $\succeq$ defined on $X = \mathbb{R}_+^2$ and that the price vector is $p \gg 0$. We have:

- ① If x^* is optimal in the VMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w .
- ② If x^* is optimal in the EMP when required utility level is $u > u(0)$, then x^* is optimal in the VMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this VMP is exactly u .

Hicksian/Compensated Demand Function

The set of optimal bundles in the EMP is denoted $h(p, u)$, Hicksian/Compensated demand correspondence.

Hicksian/compensated demand function is single valued.

Proposition 4-3-6. Suppose that $u(\cdot)$ is a continuous utility function representing a monotone preference relation $\succeq$ defined on $\mathbb{R}_+^2$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possess the following properties:

- ① Homogeneity of degree zero in p : $h(\alpha p, u) = h(p, u)$ for any p, u and scalar $\alpha > 0$.
- ② No excess utility: $u(x) = u$ for all $x \in h(p, u)$.
- ③ Convexity/Uniqueness: If $\succeq$ is convex, so that $u(\cdot)$ is quasiconcave, then $h(p, u)$ is a convex set. Moreover, if $\succeq$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $h(p, u)$ consists of a single element.

This proposition parallels Proposition on the properties of Walrasian Demand.

Proof. Property ①: Homogeneity of degree zero in p

Exam

The optimal consumption bundle when minimizing $p \cdot x$ subject to $u(x) \geq u$ is the same as that for minimizing $\alpha p \cdot x$ subject to $u(x) \geq u$, for any $\alpha > 0$.

Proof. Property ②: No Excess utility

Exam

This property follows from the continuity of $u(\cdot)$. Suppose there exists an $x \in h(p, u)$ such that $u(x) > u$. Consider a bundle $x' = \alpha x$, where $\alpha \in (0, 1)$.

By continuity, for α close enough to 1, $u(x') > u$ and $p \cdot x' < p \cdot x$. Contradicts x being optimal in the EMP.

Proof. Property ③: Convexity/Uniqueness

The proof of this property parallels that for the convexity/Uniqueness property of Walrasian demand function. Hence, we forego this proof. (May be covered in the exam.)

Exam

Expenditure Function

- Given $p \gg 0$ and required utility level $u > u(0)$, the value of EMP is denoted $e(p, u)$.
- The function $e(p, u)$ is called the **expenditure function**.
- Its value for any (p, u) is simply $p \cdot x^*$, where x^* is the solution of EMP.

Proposition 4-3-7. Suppose that $u(\cdot)$ is a continuous utility function representing a monotone preference relation $\succsim$ defined on $\mathbb{R}_+^2$. Then the expenditure function is:

- ① Homogeneity of degree one in p : $e(\alpha p, u) = \alpha e(p, u)$ for any p, u and scalar $\alpha > 0$
- ② Strictly increasing in u and nondecreasing in p_1 and p_2 .
- ③ Concave in p .
- ④ Continuous in p and u .

The four properties parallel the properties of the indirect utility function for UMP.

Proof. Property ①: Homogeneity of degree 1

The constraint set of the EMP is unchanged when prices change. Thus, for any scalar $\alpha > 0$, minimizing $(\alpha p) \cdot x$ on this set leads to the same optimal consumption bundles as minimizing $p \cdot x$. Letting x^* be optimal in both circumstances, we have $e(\alpha p, u) = \alpha p \cdot x^* = \alpha e(p, u)$

Exam

Proof. Property ②. Part I: Increasing in u .

Suppose that $e(p, u)$ were not strictly increasing in u , and let x' and x'' denote optimal consumption bundles for required utility levels $u' < u''$, respectively. Then we have $p \cdot x' \geq p \cdot x''$. Consider a bundle $\bar{x} = \alpha x''$ where $\alpha \in (0, 1)$.

By continuity of $u(\cdot)$, there exists α close enough to 1 such that $u(\bar{x}) > u'$ and $p \cdot x' > p \cdot \bar{x}$.

But this contradicts x' being optimal in the EMP with required utility level u' .

Part II: Nondecreasing in p .

Suppose that price vectors p'' and p' have $p''_1 \geq p'_1$ and $p''_2 = p'_2$. Let x'' be an optimizing bundle in the EMP for prices p'' . $e(p'', u) = p'' \cdot x'' \geq p' \cdot x'' \geq e(p', u)$. The latter inequality follows from the definition of $e(p', u)$.

Exam

Proof. Property ③: Concavity in p

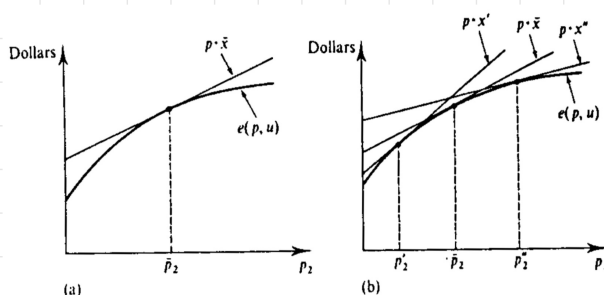
Fix a required utility level $\bar{u}$. Let $p'' = \alpha p + (1 - \alpha)p'$ for $\alpha \in [0, 1]$.

Suppose that x'' is an optimal bundle in the EMP for prices p'' . Then $e(p'', u) = p'' \cdot x'' = \alpha p \cdot x'' + (1 - \alpha)p' \cdot x'' \geq \alpha e(p, \bar{u}) + (1 - \alpha)e(p', \bar{u})$,

where the last inequality follows because $u(x'') \geq \bar{u}$ and the definition of the expenditure function implies that $p \cdot x'' \geq e(p, \bar{u})$ and $p' \cdot x'' \geq e(p', \bar{u})$.

The concavity of $e(p, \bar{u})$ in p for given $\bar{u}$ is actually fairly intuitive. Suppose that we initially have $\bar{p}$ and that $\bar{x}$ is an optimal bundle at $\bar{p}$. If prices change but we do not let the consumer change her consumption bundle, then the resulting expenditure will be $p \cdot \bar{x}$, which is a linear expression in p . But, when the consumer can adjust her consumption, as in the EMP, her minimized expenditure can be no greater than this amount. This amounts to concavity because a similar relation must hold at each point of the graph of $e(\cdot, \bar{u})$.

Exam



Example: EMP with Cobb-Douglas utility function. $\min_{x \geq 0} p_1 x_1 + p_2 x_2$ s.t. $x_1^\alpha x_2^{1-\alpha} \geq u \Rightarrow \min_{x \geq 0} p_1 x_1 + p_2 x_2$ s.t. $x_1^\alpha x_2^{1-\alpha} = u$

$$\Rightarrow \min_{x \geq 0} p_1 x_1 + p_2 u^{1/(1-\alpha)} x_1^{-\alpha/(1-\alpha)} \Rightarrow \text{FOC: } p_1 + p_2 u^{1/(1-\alpha)} \frac{-\alpha}{1-\alpha} x_1^{-1-\alpha/(1-\alpha)} = 0 \Rightarrow p_1 x_1^{1/(1-\alpha)} = p_2 u^{1/(1-\alpha)} \frac{\alpha}{1-\alpha} \Rightarrow$$

$$\Rightarrow x_1 = \left[\frac{\alpha p_2}{(1-\alpha) p_1} \right]^{1-\alpha} u > 0 \quad \text{then } h_1(p, u) = \left[\frac{\alpha p_2}{(1-\alpha) p_1} \right]^{1-\alpha} u, \quad h_2(p, u) = \left[\frac{(1-\alpha) p_1}{\alpha p_2} \right]^\alpha u$$

$$\text{and } e(p, u) = [\alpha^\alpha (1-\alpha)^{1-\alpha}] p_1^\alpha p_2^{1-\alpha} u.$$

Exercise: Verify the properties of $h(p, u)$ and $e(p, u)$.

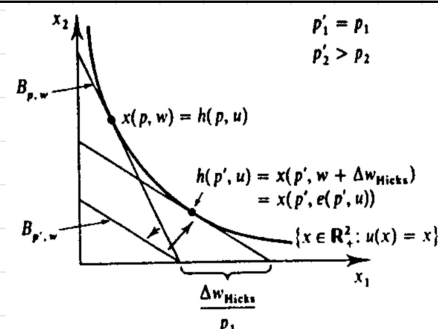
VMP and EMP

Link between Hicksian demand and Walrasian demand: $h(p, u) = x(p, e(p, u))$ and $x(p, w) = h(p, v(p, w))$.

Compensated demand: An price vary, $h(p, w)$ gives precisely the level of demand that would arise if the consumer's wealth were simultaneously adjusted to keep her utility level at u .

This type of wealth compensation is known as Hicksian wealth compensation. $\Delta w_{\text{Hicks}} = e(p', u) - w$.

Thus, the demand function $h(p, u)$ keeps utility level fixed as prices changes. Walrasian demand keeps money wealth fixed but allows utility to vary.



Hicksian Demand and Compensated law of demand

Proposition 4-3-8 Suppose that $u(\cdot)$ is a continuous utility function representing a monotone preference relation $\succsim$ defined on $\mathbb{R}_+^2$ and that $h(p, u)$ consists of a single element for all $p \gg 0$. Then $h(p, u)$ satisfies the compensated law of demand: For all p' and p'' , $(p'' - p') [h(p'', u) - h(p', u)] \leq 0$.

- Own-price effects are nonpositive.
- The comparable statement is not true for Walrasian demand.
- Walrasian demand need not satisfy the law of demand. Giffen goods.

Proof: For any $p \gg 0$, consumption bundle $h(p, u)$ is optimal in the EMP, so it achieves a lower expenditure at prices p than any other bundle that offers a utility level of at least u . Therefore, we have $p'' \cdot h(p'', u) \leq p'' \cdot h(p', u)$ and $p' \cdot h(p'', u) \geq p' \cdot h(p', u)$. Subtracting these two inequalities yields the result.

Exam

A Jungle

Topic 4 studies one individual's decision making in the market. We now, for the first time, consider an entire economy in which individuals interact.

A Jungle game

There are 4 individuals, $\{1, 2, 3, 4\}$ and 4 caves, $\{A, B, C, D\}$ in a jungle. Each cave can accommodate only one person and each person can occupy only one cave. The strength order is $1 > 2 > 3 > 4$. A person who wants to occupy a cave currently occupied by a weaker person can do so.

The preferences are the following:

Individuals			
1	2	3	4
B	B	A	B
C	D	B	C
D	A	C	A
A	C	D	D

Model: Society

A society consisting of a set of individuals and a set of houses. Each house can accommodate only one person and each person can occupy only one house. Different people may have different preferences over the houses, but everyone prefers to occupy any house than to be homeless.

The assignment of houses is determined by the individuals' strength; the concepts of property and ownership do not exist.

A person who wants to occupy a house currently occupied by a weaker person can do it simply by presenting himself to the current occupant.

The process is orderly: everyone knows everyone else's strength, and on seeing that a stronger person wants to occupy his house, a person vacates it without a fight, which he knows he would lose.

Definition 5-1-1: Society

A society $\langle N, H, (\succsim^i)_{i \in N} \rangle$ consists of:

- ① Individuals, a finite set N .
- ② Houses, a finite set H with the same number of members as N .
- ③ Preferences, a strict preference $\succsim^i$ for each individual $i \in N$.

Definition 5-1-2: Assignment and Allocation

An assignment for a society $\langle N, H, (\succsim^i)_{i \in N} \rangle$ is a function from the set N of individuals to the set H of houses, associating a house with every individual. An allocation is an assignment in which each house is assigned to exactly one individual.

What determines the allocation of houses? Presumably the individuals' preferences play a role.

What if two individual both want one house? An additional factor: the **individuals' relative power**.

- We assume that between any two individuals there is a stable power relation: either i is stronger than j , or j is stronger than i .
- No two individuals are equally powerful.
- Precisely, we take as given a binary relation $\triangleright$ on the set of individuals, where $i \triangleright j$ means that i is more powerful than j , in which case i can take over the house occupied by j .

Definition 5-1-3: Jungle

A jungle $\langle N, H, (\succsim^i)_{i \in N}, \triangleright \rangle$ consists of a society $\langle N, H, (\succsim^i)_{i \in N} \rangle$ and a power relation $\triangleright$, a complete, transitive, antisymmetric binary relation on the set N of individuals.

Remarks: we assume that the number of houses is equal to the number of individuals and that no individual is indifferent between any two houses

- Power is not only physical
- No property rights.
- Outcome of a struggle is deterministic
- No cost of fighting.

Equilibrium

An equilibrium in a model is generally an outcome that is stable given the forces assumed to be active. In a jungle, a stable outcome is an allocation in which no individual prefers any house that he can obtain to the one he occupies. Thus, an allocation is not an equilibrium if some individual prefers the house occupied by a weaker individual to the house he currently occupies.

Definition 5-1-4: Equilibrium

An equilibrium of the jungle $\langle N, H, (\succsim^i)_{i \in N}, \triangleright \rangle$ is an allocation a^* such that for no individuals $i, j \in N$ is it the case that $i \triangleright j$ and $a^*(j) \succ^i a^*(i)$.

An equilibrium of a jungle is an allocation a for which no individual i prefers the house $a(j)$ occupied by any individual j who is weaker than i to the house $a(i)$ that she occupies.

Example, for the jungle consisting of the society we introduced earlier together with the power relation $\triangleright$ for which $1 \triangleright 2 \triangleright 3 \triangleright 4$, the allocation (B, D, A, C) is an equilibrium.

Equilibrium: Existence

Every jungle has an equilibrium. That is, for every set of individuals, set of houses, profile of preference relations, and power relation, an allocation - at least one, possibly more than one - is stable against the exercise of power. We prove this result by showing that an allocation generated by a procedure called serial dictatorship is an equilibrium.

Equilibrium: Serial Dictatorship

Definition 5-1-5: Serial Dictatorship

For the society $\langle N, H, (\succsim^i)_{i \in N} \rangle$ and ordering $i_1, i_2, \dots, i_n$ of the members of N , the serial dictatorship procedure generates the allocation defined inductively as follows. **Initialization:** the house h_1 allocated to individual i_1 in the best house in H according to i_1 's preference relation, $\succsim^{i_1}$.

Inductive Step: For every $k \geq 2$, the house h_k allocated to individual i_k in the best house in $H \setminus \{h_1, \dots, h_{k-1}\}$ according to i_k 's pref. relation, $\succsim^{i_k}$.

Proposition 5-1-1: Every jungle has an equilibrium

For any jungle $\langle N, H, (\succsim^i)_{i \in N}, \triangleright \rangle$ and the allocation generated by the serial dictatorship procedure for the society $\langle N, H, (\succsim^i)_{i \in N} \rangle$ and the ordering $\triangleright$ is an equilibrium.

Proof: Let a be the assignment generated by the serial dictatorship procedure for the society $\langle N, H, (\succsim^i)_{i \in N} \rangle$ and the ordering $\triangleright$.

The number of houses in the name as the number of individuals, so the procedure assigns to every individual a house. Every house is assigned only once, so the assignment a is feasible, and hence an allocation. The allocation is an equilibrium because the house assigned to each individual is the best house, according to the individual's preferences, among all the houses not assigned to stronger individuals.

Exam

Proposition 5-1-2: Every jungle has a unique equilibrium

Every jungle has a unique equilibrium.

Proof: (By Contradiction) Without loss of generality assume that $N = \{1, 2, \dots, n\}$ and $1 \triangleright 2 \triangleright \dots \triangleright n$.

Assume that a and b are two equilibria of the jungle.

Denote by i^* the strongest individual i for whom $a(i) \neq b(i)$. That is, $a(i) = b(i)$ for all $i < i^*$ and $a(i^*) \neq b(i^*)$.

$i:$	1	2	...	i^*-1	i^*	...	j	...	k	...	n
$a(i):$	red	green	...	red	orange	...	grey	...	blue	...	grey
$b(i):$	red	green	...	red	blue	...	orange	...	grey	...	grey

a is an equilibrium $\Rightarrow a(i^*) \succ^{i^*} a(k) = b(i^*)$;
 b is an equilibrium $\Rightarrow b(i^*) \succ^{i^*} b(j) = a(i^*)$. $\searrow$ contradiction.

The concept of equilibrium in a jungle is static.

• Problem 2 (b) in PS5 describes a dynamic process that starts from an arbitrary initial allocation and asks you to show that this process converges to an equilibrium.

The assumption that the individuals' preferences are strict is essential for this result.

• Problem 2 (a) in PS5 asks you to show that if one or more individuals are indifferent between two houses then a jungle may have more than one equilibrium.

A Clan Game

There are 4 individuals, $\{1, 2, 3, 4\}$ and 4 caves, $\{A, B, C, D\}$. Each cave can accommodate only one person and each person can occupy only one cave. The allocation of houses to people is determined not by the balance of power, but by mutual agreement. One allocation is replaced by another only if everyone agrees: no one objects and at least one person prefers the new allocation.

The preferences are the following:

Individuals			
1	2	3	4
B	B	A	B
C	D	B	C
D	A	C	A
A	C	D	D

Pareto Stability

The game ends when the current allocation is immune to replacement.

No allocation is better for some people and no worse for anyone.

Definition 5-1-6: Pareto Stability

The allocation b in the society $\langle N, H, (\succsim^i)_{i \in N} \rangle$ Pareto dominates the allocation a if $b(i) \succsim^i a(i)$ for all $i \in N$ and $b(i) \succ^i a(i)$ for some $i \in N$. An allocation is Pareto stable if no allocation Pareto dominates it.

Pareto Optimal = Pareto Efficient = Pareto Stable The notion of Pareto stability does not involve power. Another potential source of instability. Whenever one allocation is Pareto dominated by another, the latter will replace the former. What if both b and c dominate a ?

Proposition 5-1-3: Serial dictatorship allocation is Pareto Stable

For any society and any ordering of the individuals, the allocation generated by the serial dictatorship procedure is Pareto Stable.

Proof:

$i:$	i_1	i_2	$\dots$	i_{r-1}	i_r	$\dots$	i_s	$\dots$	i_n
$a(i):$	▲	■	$\dots$	■	■	$\dots$	■	$\dots$	■
$b(i):$	▲	■	$\dots$	■	■	$\dots$	■	$\dots$	■

Equilibrium and Pareto Stability in a jungle

We now connect the notion of equilibrium and Pareto stability.

Equilibrium: an outcome be immune to the use of power.

Pareto Stability: the outcome be immune to a reallocation to which everyone agrees.

Proposition 5-1-4: The equilibrium of every jungle is Pareto stable.

Proof: Every jungle has a unique equilibrium. This equilibrium is obtained by applying the serial dictatorship procedure for the power relation. Serial dictatorship allocation is Pareto Stable.

An Exchange Economy

- Topic 4 studies one individual's decision making in the market.
- Topic 5-1 consider an entire economy in which individuals interact in a jungle or a clan.
- We now consider an entire economy in which individuals interact through **markets**.

A market game

There are two individuals, {Ann, Bob} and two goods {Apple, Banana}, in an island. Ann has 6 apples, and Bob has 6 bananas. Ann and Bob's preferences are represented by $u_1(x_1, x_2) = x_1^{2/3} x_2^{1/3}$, $u_2(x_1, x_2) = x_1^{1/3} x_2^{2/3}$ respectively. Ann and Bob can trade with each other.

$$\text{Ann} = \left(\frac{2}{3} \frac{w_1}{P_1}, \frac{1}{3} \frac{w_1}{P_2}\right) \text{ and } \text{Bob} = \left(\frac{1}{3} \frac{w_2}{P_1}, \frac{2}{3} \frac{w_2}{P_2}\right) \text{ where } w_1 = 6P_1 \text{ and } w_2 = 6P_2 \text{ then Ann} = \left(4, \frac{2P_2}{P_1}\right) \text{ and Bob} = \left(\frac{2P_2}{P_1}, 4\right)$$

Model

Definition 5-2-1: Exchange Economy

An exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$ consists of:

- ① **individuals**, a finite set N .
- ② **preferences** for each individual $i \in N$, a monotone and continuous preference relation $\succeq^i$ over $\mathbb{R}_+^2$.
- ③ **initial allocation**, a function e that assigns to each individual $i \in N$ a bundle $e(i) \in \mathbb{R}_+^2$, the bundle that i initially owns, with $\sum_{i \in N} e_k(i) > 0$ for $k=1,2$.

Price System and Budget Set

Definition 5-2-2: Price System and Budget Set

A price system is a pair of nonnegative numbers different from $(0,0)$. The value of the bundle $x = (x_1, x_2)$ according to the price system $p = (p_1, p_2)$ is $p \cdot x = p_1 x_1 + p_2 x_2$, and for the exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$, budget set of individual $i \in N$ given the price system p is $B(p, e(i)) = \{x \in \mathbb{R}_+^2 : p \cdot x \leq p \cdot e(i)\}$.

Price System

Two interpretations of a price system. The first and most literal is that prices are quoted in a monetary unit. Each individual can sell any amounts of the goods in his initial bundle and use the monetary proceeds to buy amounts of other goods. The second interpretation is that the prices represent the ratio at which the goods may be exchanged. Specifically, the price system (p_1, p_2) means that one unit of good 1 may be exchanged for p_1/p_2 units of good 2.

In both interpretations, a price system p is equivalent to any price system of the form $\lambda p = (\lambda p_1, \lambda p_2)$ for $\lambda > 0$.

Allocations

Definition 5-2-3: Assignment and Allocation

An assignment in an exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$ is a function from the set N of individuals to the set of $\mathbb{R}_+^2$ of possible bundles of goods. An allocation is an assignment a for which the sum of the assigned bundles is the sum of the initial bundles: $\sum_{i \in N} a(i) = \sum_{i \in N} e(i)$.

Competitive Equilibrium

Definition 5-2-4: Competitive Equilibrium

A competitive equilibrium (CE) of the exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$ is a pair (p, a) in which: $P = (p_1, p_2)$ is a price system and a is an assignment, such that:

- ① **Optimality of choices** for every individual $i \in N$ the bundle $a(i)$ is optimal according to $\succeq^i$ in the budget set $B(p, e(i))$.

② **Feasibility**, a is an allocation.

An allocation is a competitive equilibrium allocation of the exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$ if for some price system p , (p, a) is a competitive equilibrium of $\langle N, (\succeq^i)_{i \in N}, e \rangle$.

- When is (p, a) not a CE? Either at least one individual does not choose her optimal bundle or for at least one of the goods, the total amount that the individuals want to purchase is different from the total amount available.
- If (p, a) is a CE then so is $(\lambda p, a)$ for any $\lambda > 0$.
- In a CE all individuals face the same price system and each individual chooses an optimal bundle from a budget set defined by this price system, which is not affected by the individual's choice, seems reasonable when N is large.
- Is well-defined when $N=1$? Yes, it is. Only the reasonableness of the concept is questionable.

Proposition 5-2-1: Property of assignment of bundles on budget lines

Consider an exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$. Let p be a price system with $(p_1, p_2) \gg 0$. Consider an assignment a with $pa(i) = pe(i)$ for all $i \in N$. If the sum of the quantities of one good in the bundles in the assignment a is equal to the sum of the quantities of the good in the initial bundles then this equality holds also for the other good. That is,

$$\sum_{i \in N} a_1(i) = \sum_{i \in N} e_1(i) \iff \sum_{i \in N} a_2(i) = \sum_{i \in N} e_2(i).$$

Given any price system and any assignment that consists of optimal bundles (given the price system), if for one good the sum of the assigned quantities is equal to the sum of the individuals' initial holdings, then the same is true also for the other good. This result is useful when calculating competitive equilibria, because it means that if we find a price system for which demand and supply are equal for one good, then we know that they are equal for the other good, so that the price system is consistent with competitive equilibrium. How about an exchange economy with L goods? If $L-1$ markets clear, then all markets clear.

Proof: By $pa(i) = pe(i)$ for each i , we have: $\sum_{i \in N} [p_1 a_1(i) + p_2 a_2(i)] = \sum_{i \in N} [p_1 e_1(i) + p_2 e_2(i)]$, so $p_1 \left(\sum_{i \in N} a_1(i) - \sum_{i \in N} e_1(i) \right) = p_2 \left(\sum_{i \in N} a_2(i) - \sum_{i \in N} e_2(i) \right)$.

So that, given $p_1 > 0$ and $p_2 > 0$, $\sum_{i \in N} a_1(i) - \sum_{i \in N} e_1(i) = 0 \iff \sum_{i \in N} a_2(i) - \sum_{i \in N} e_2(i) = 0$.

Proposition 5-2-3: Pareto Stability of Competitive Equilibrium Allocation

Every competitive equilibrium allocation of an exchange economy is Pareto stable.

The name conventionally given to this result and similar results for other models of economies is the "First Fundamental Theorem of Welfare Economics". The result says that for any equilibrium allocation, no other allocation exists for which some individual is better off and no individual is worse off. The invisible hand is (Pareto) efficient. An implication is that if the initial allocation is not Pareto stable, every equilibrium involves exchange/trade. The converse statement is also true: if the initial allocation is Pareto stable, then no trade occurs in equilibrium.

The conclusion of this result depends critically on the assumption that each individual cares only about the bundle he consumes (No Externality). Consider a variant of an exchange economy in which individuals care also about the bundles consumed by other individuals. Suppose specifically that the economy contains two individuals, with initial bundles $(1,0)$ and $(0,1)$. Individual 1 is negatively affected by individual 2's consumption of good 2; his utility is $a_1(1) + a_2(1) - 2a_2(2)$. Individual 2 cares only about his own consumption; his utility is $2a_1(2) + 3a_2(2)$. Assuming that each individual takes the consumption of the other individual as given, $(1,1)$ and no trade (the induced allocation is the initial allocation) is a CE. This outcome is not Pareto Stable: the allocation b for which $b(1) = (0, 0.5)$ and $b(2) = (1, 0.5)$ is preferred by both individuals.

Proof: Let (p, a) be a CE of $\langle N, (\succeq^i)_{i \in N}, e \rangle$. Assume that a is not Pareto Stable. Then there is an allocation y such that $y(i) \succeq^i a(i)$ for all i and $y(j) \succ^j a(j)$ for some j . $a(i)$ is optimal according to $\succeq^i$ in i 's budget set implies $py(i) \geq pe(i)$. Otherwise, $a(i)$ is not optimal. $a(j)$ is optimal implies $py(j) > pe(j)$. Otherwise, $a(j)$ is not optimal. Thus, $p \cdot \sum_i y(i) > p \cdot \sum_i e(i)$. Contradicting the feasibility of y , $\sum_i y(i) > \sum_i e(i)$. Hence, a is Pareto Stable.

Game Theory

Introduction

The first half we dealt with **Decision Theory**, it analyzes the decision problems of an individual in isolation. Now, in the second half, we deal with **Game Theory**, it can be called also **Interactive Decision Theory**.

Game Theory (GT) is the formal analysis of the behaviour of interacting individuals. The crucial feature of an interactive situation is that **the consequences of the actions of an individual depend (also) on the actions of other individuals: Interdependence**.

This is typical of many games people play for fun, such as chess or poker. Hence, interactive situations are called "games" and interactive individuals are called "players". If a player's behaviour is intentional and he is aware of the interaction, he should try and anticipate the behaviour of other players. This is the essence of **Strategic Thinking**.

We try to develop an understanding of how people actually behave, and how they should be advised to behave, in strategic situations. First, it identifies a language with which we can converse and exchange ideas about human endeavour. Second, it provides a framework that guides us in constructing models of strategic settings - a process that engenders insights by challenging us to be clear and rigorous in our thinking. Third, it helps us trace through the logical implications of assumptions about behaviour.

Four main topics: **Representations and Basic Assumptions, Static Games, Dynamic Games, Information**.

Representations

There are several different ways of describing games mathematically. The representations presented in this course have the following formal elements in common:

1. A list of **players**
2. A complete description of **what players can do** (their possible actions).
3. A description of **what players know** when they act.
4. A specification of **how the players' action lead to outcomes**.
5. A specification of the **players' preferences over outcomes**.

There are two common forms in which (noncooperative) games are represented mathematically: the **extensive form** and the **normal (strategic) form**.

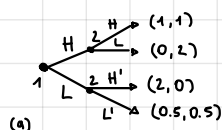
Extensive Form

We start with a very simple example, a price-competition game. Two firms (1 and 2) compete by each selecting either a High (H) or Low (L) price for a product that they both produce.

Firm 1 selects price (either H or L) first. Observing Firm 1's selection, Firm 2 selects price.

The monetary payoffs are the following: if both firms select H, then they share the market, each gets 1 M; if one selects H and the other selects L, then low price firm takes all the market and gets 2 M, the high price firm gets 0; if both select L, each gets 0.5M.

This situation can be analysed as a game which can be represented with a rooted tree with utility numbers attached to terminal nodes (leaves), player labels attached to non-terminal nodes, and action labels attached to arcs/branches.



Nodes represent places where something happens in the game (such as a decision by one of the players), and branches indicate the various actions that players can choose.

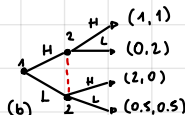
We list the players' payoffs in the order of the players' identifying numbers. Player 1's payoff is written first, followed by Player 2's payoff, and so on.

Now we consider another price-competition in which each firm has to select between H or L without knowing the other firm's decision.

The players learn about each other's choices only after both are made.

Referring to the tree in (a), we represent Firm 1's lack of information by specifying that, during the game, it cannot distinguish between two nodes in which it makes decisions.

How to represent this lack of information? Adding a dashed line.



Decision Notes: Players make decisions at these places in the game.

Terminal Notes: Represent outcomes of the game — places where the game ends.

Each terminal node also corresponds to a unique path through the tree, which is a way of getting from the initial node through the tree by following branches in the direction of the arrows.

In an extensive form, there is a **one-to-one relation** between paths and terminal nodes.

It is common to use the term **Information Set** to specify the players' information at decision nodes in the game.

An **Information Set** describes which decision nodes are connected to each other by dashed lines (meaning that a player cannot distinguish between them). Info sets **precisely** describe the **different** decisions that players have to make.

Every decision node is contained in an information set; some information sets consist of only one node. The information sets in a game precisely describe the different decisions that players have to make: x info sets, x decisions to make.

How many info sets for Firm 2 in (a)? 2

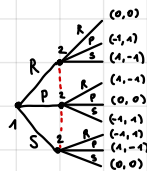
How many decisions to make for Firm 2 in (a)? 4

How many info sets for Firm 2 in (b)? 1

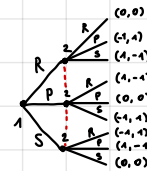
How many decisions to make for Firm 2 in (b)? 4

Extensive Form Exercise

Two players simultaneously throw their right arms up and down to the count of "one, two, three". (Nothing strategic happens as they do this). On the count of three, each player quickly forms his or her hand into the shape of either a rock (R), a piece of paper (P), or a pair of scissors (S). The players make this choice at the same time. If the players pick the same shape, then the game ends in a tie. Otherwise, one of the players wins and the other loses. The winner is determined by the following rule: rock beats scissors, scissors beats paper, and paper beats rock. Each player obtains a payoff of 1 if he or she wins, -1 if he or she loses, and 0 if he or she ties. Represent this game in the extensive form. Also discuss the relevance of the order of play (which of the player has the move at the initial node) in the extensive form.



If player 2 was to move after player 1's decision then we would simply remove the dashed line.



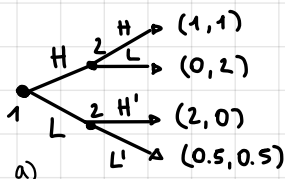
Definition: Strategy

A **Strategy** is a complete contingent plan for a player in the game.

One of the most important concepts in game theory. The "**complete contingent**" part of this definition is what makes it powerful and, to some people, a bit confusing.

A full specification of a player's behaviour, which describes the actions that the player would take at each of his possible decision points.

Information sets represent places in the game at which players make decisions, a **player's strategy** describes what he will do at each of his information sets.



Player's 2 Strategy tells us what this player will do at his top node (an information set) and at his bottom node (another information set). In words, in this game, a strategy for player 2 specifies what price to pick in response to H selected by player 1 as well as what price to pick in response to player's 1 selection of L.

How to write Strategies? The easiest way of writing strategies is to put together the labels corresponding to the actions to be chosen at each information set. For example, one strategy for player 2 is to select the high price (H) in the top information set and the low price (L) in the bottom of information set; this strategy can be written HL'.

There are four strategies for player 2: HH', HL', LH' and LL'. We use different labels for a player's various information sets to avoid confusions.

Strategy Space

Given a game, we let S_i denote the **Strategy Space** (also called the **Strategy Set**) of player i . That is, S_i is a set comprising each of the possible strategies of player i in the game. For the game shown in figure a), the strategy space of player 1 is $S_1 = \{H, L\}$, and the strategy space of player 2 is $S_2 = \{HH', HL', LH', LL'\}$. We use lowercase letters to denote single strategies (generic members of these sets).

Thus, $s_i \in S_i$ is a strategy for player i in the game. We could thus have $s_1 = L$ and $s_2 = LH'$, for instance.

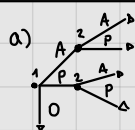
A **Strategy Profile** is a vector of strategies, one for each player. In other words, a strategy profile describes strategies for all of the players in the game. For example, suppose we are studying a game with n players. A typical strategy profile then is a vector $s = (s_1, s_2, \dots, s_n)$, where s_i is the strategy of player i , for $i = 1, 2, \dots, n$.

Let S denote the set of strategy profile. Mathematically, we write $S = S_1 \times S_2 \times \dots \times S_n$. Note that the symbol "x" denotes the Cartesian product.

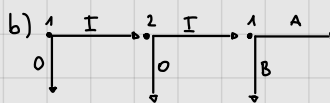
For example, if $S_1 = \{A, B\}$ and $S_2 = \{X, Y\}$, then $S = S_1 \times S_2 = \{(A, X), (A, Y), (B, X), (B, Y)\}$.

$S_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.

Examples



$S_1 = \{O, A, P\}$
 $S_2 = \{A, P\}$



$S_1 = \{IA, IB, OA, OB\}$
 $S_2 = \{I, O\}$

Complete contingent plan means more than just a "plan" for how to play the game. Player 1 in the game shown in Figure (b) needs a plan for his second information set if he selects O at the first! Can't we just say that player 1 has three different strategies, O, IA, and IB? In fact, no.

The definition of a strategy (a complete contingent plan) requires a specification of player 1's choice at his second information set even in the situation in which he plans to select O at his first information set. Stating that player 1's plan is "O" does NOT provide enough information for us to conduct a thorough analysis.

The study of rationality will explicitly require the evaluation of players' optimal moves starting from arbitrary points in a game. This evaluation is connected to the beliefs that players have about each other. Player 1's optimal choice at his first information set depends on what he thinks player 2 would do if put on the move. Furthermore, to select the best course of action, player 2 must consider what player 1 would do at his second information set. Thus, player 2 must form a belief about player 1's action at the third node. A belief is a conjecture about what strategy the other player is using; therefore, player 1's strategy must include a prescription for his second information set, regardless of what this strategy prescribes for his first information set.

Another reason why it is important to specify a complete contingent plan is that players may make mistakes in the course of play. For example, suppose that you are player 1 in the game shown in the previous example. Suppose that I am instructing you on how to play this game. I might write instructions on a piece of paper, hand the paper to you, and then leave town before the game begins. I could write "select O", but, even though that instruction seems to be enough information in advance, it does not give you sufficient guidance. You might make a mistake and select I instead of O at the beginning of the game. What if player 2 then picks I? There you are, red faced and indignant, yelling at the sky, "Now what am I supposed to do?"

Exercise

Two players have to decide how much to contribute to a charitable project. Suppose it is known that the project will be successful if and only if the same of the players' contributions is at least 900. The players know that they each have 600 available to contribute. The players make contributions sequentially. In the first stage, player 1 selects her contribution $x \in [0, 600]$, and player 2 observes this choice. Then, in the second stage, player 2 selects his contribution $y \in [0, 600]$. Hence, $S_1 = [0, 600]$, while player 2 has an infinite number of information sets, $S_2: [0, 600] \rightarrow [0, 600]$.

Normal Form

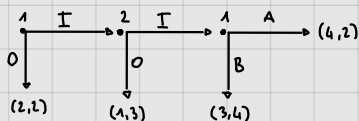
The extensive form is one straightforward way of representing a game. Another way of formally describing games is based on the idea of strategies. It is called the **normal form** (or **strategic form**) representation of a game. This alternative representation is more compact than the extensive form in some settings.

Let's start with any game in extensive form. Each strategy profile fully describes how the game is played.

Strategy Profile $\rightarrow$ a path of game tree $\rightarrow$ terminal node $\rightarrow$ payoff vector.

In short, each strategy profile implies a specific payoff vector. Define $u_i : S \rightarrow \mathbb{R}$ as player i 's payoff function. For each $s \in S$, $u_i(s)$ is player i 's payoff in the game.

Example



$$S = \{(OA, O), (OA, I), (OB, O), (OB, I), (IA, O), (IA, I), (IB, O), (IB, I)\}.$$

To determine $u_i(s)$ for any strategy profile s , simply start at the initial node and trace through the tree according to the actions specified by this strategy profile.

$$u_1(OA, O) = 2, u_1(IA, I) = 4.$$

Definition: Normal Form game

A game in normal form (also called strategic form) consists of:

- ① A set of players, $\{1, 2, \dots, n\}$
- ② Strategy spaces for the players, $S_1, S_2, \dots, S_n$.
- ③ Payoff functions for the players, $u_1, u_2, \dots, u_n$.

A convenient way of describing the strategy spaces of the players and their payoff functions for two-player games in which each player has a finite number of strategies is to draw a matrix. Each row of the matrix corresponded to a strategy of player 1, and each column corresponds to a strategy of player 2. Thus, each cell of the matrix (which designates a single row and column) corresponds to a strategy profile. Inside a given cell, we write the payoff vector associated with the strategy profile.

		2	
		I	O
1	I	2, 2	2, 2
	O	2, 2	2, 2
	IA	4, 2	1, 3
	IB	3, 4	1, 3

Extensive Form and Normal Form

There may be only one way of going from the extensive form to the normal form, the reverse is not true.

Classic Normal-Form Games

The figure displays nine 2x2 payoff matrices for different game theory scenarios. Each matrix is labeled with a title and shows the payoffs for two players (1 and 2) based on their choices.

Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Prisoners' Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Battle of the Sexes

	Opera	Movie
Opera	2, 1	0, 0
Movie	0, 0	1, 2

Hawk-Dove/Chicken

	H	D
H	0, 0	3, 1
D	1, 3	2, 2

Coordination

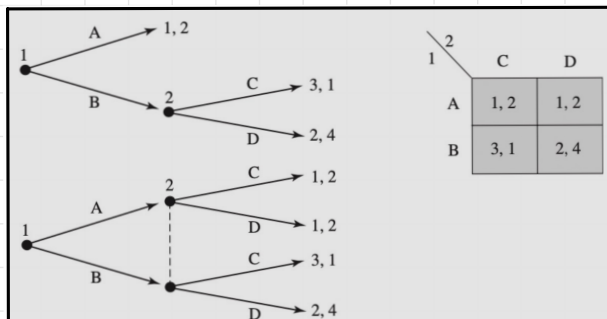
	A	B
A	1, 1	0, 0
B	0, 0	1, 1

Pareto Coordination

	A	B
A	2, 2	0, 0
B	0, 0	1, 1

Pigs

	P	D
P	4, 2	2, 3
D	6, -1	0, 0



Belief, Mixed Strategy & Expected Utility

Belief

Suppose we are playing the Battle of the Sexes (BoS) game. You are player 1 and I am player 2.

1 \ 2	A	B
A	1, 3	0, 0
B	0, 0	3, 1

Suppose you think I will surely play B. What strategy will you choose? B

Suppose you think I will surely play A. What strategy will you choose? A

Suppose you are not sure about my strategy. What strategy will you choose? A

It is important for players to think about each other's strategic choices. We use the term belief for a player's **answer** about the strategies of the others in the game.

Definition: Belief

A belief of player i , $\theta_i \in \Delta S_{-i}$, is a probability distribution over the strategies of the other players, where ΔS_{-i} is the set of probability distribution over the strategies of all the players except player i .

What does $\sigma_1 = (1/3, 2/3)$ mean? Player 1 might not believe that player 2 **actually** randomizes (by say, flipping a coin).

It's just that player 1 may be uncertain about what strategy player 2 will choose, and so he associates probabilities with player 2's strategies.

Furthermore, player 1's belief may not be accurate; he could be certain that player 2 will select A when in fact player 2 selects B.

Is σ_1 subjective or objective? Subjective.

Exercise

Pick a larger number game. Player 1 and 2 simultaneously announce a number between 0 and 1. The player with the larger number is the winner. Each player obtains a payoff of 1 if he wins, -1 if he loses, and 0 if he ties. What are the strategy spaces? $S_i = [0, 1]$. Find one belief of player 1. It could, for example, be a uniform distribution on the set $[0, 1]$. What is the set of all possible beliefs of player 1, S_{-1} ? The set of all probability distributions on $[0, 1]$.

Mixed Strategy

Related to a belief is the notion of a mixed strategy. A mixed strategy for a player is the act of selecting a strategy according to a probability distribution. She might flip a coin and select A if the coin lands with the head up and B if the coin lands with the tail up. A mixed strategy of player i , $\sigma_i \in \Delta S_i$, is a probability distribution over his strategy space, where ΔS_i is the set of probability distribution over his strategy space.

Expected Payoff

1 \ 2	A	B
A	1, 3	0, 0
B	0, 0	3, 1

Suppose $\sigma_1 = (1/2, 1/2)$ and $\theta_{-1} = (1, 0)$. What is the expected payoff? $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = 1/2$

Suppose $\sigma_1 = (1, 0)$ and $\theta_{-1} = (1/2, 1/2)$. What is the expected payoff? $1 \cdot 1/2 + 0 \cdot 1/2 = 1/2$

Suppose $\sigma_1 = (1/2, 1/2)$ and $\theta_{-1} = (1/2, 1/2)$. What is the expected payoff? $\frac{1}{2} \cdot \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2} \cdot 3 = \frac{1}{4} + \frac{3}{4} = 1$

In general, what is your expected payoff under σ_1 and θ_{-1} ? $u_i(\sigma_i, \theta_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i) \theta_{-i}(s_{-i}) u_i(s_i, s_{-i})$.

In $u_i(\sigma_i, \theta_{-i})$ expected utility or subjective utility? Subjective.

General Assumptions

Rationality

Game Theory is about rational choice of strategies when each agent's utility depends not only on their own choices, but also on the choices of other agents. We assume that, if a player's action will determine which of several outcomes will occur in a game, then this player will select the action that leads to the outcome he most prefers. Rationality means maximising one's subjective expected payoff.

Rationality is a weak assumption in the sense that any coherent preferences can be accommodated by the theory. Rationality does not necessarily imply that the players seek to maximise their own monetary gains, though we often assume it in the course. In some settings, however, players are highly motivated by other factors. It is important to accurately describe their preferences by including the non traditional elements in the description of payoffs. The key point to remember is that game theory can accommodate any preferences that have a mathematical representation.

Common Knowledge

Conventional analysis of behaviour requires that players have **shared understanding of the entire game**. The players know the extensive-form (or normal-form) game that they are playing, which includes the recognition that, in the course of play, individual players will not be able to distinguish between nodes in the same information set. **Shared understanding** is vague. To be precise, we use the notion of **Common Knowledge**.

Definition: Common Knowledge

A particular fact F is said to be common knowledge between the players if each player knows F , each player knows that the others know F , each player knows that every other player knows that each player knows F , and so on.

We assume that **the game is common knowledge between the players**. It is as if, before the game is played, the players gather around a table on which the game has been depicted.

A Puzzle

Imagine three professors, A, B, C, sitting in a circle, each wearing either a red hat or a white hat (as a Bocconi faculty meeting tradition). Each professor can see the hats of the other two professors, but not his own.

Suppose that all the hats are red.

Then Professor D asks the question of the day, "What's the color of your hat today?" No Professor could answer this question.

D certainly is not satisfied with their answers and says "everyone knows that there is, at least, one red hat in the room, and we are professors, we could do better than this!"

After announcing this well-known to everyone fact, D asks again. "A, what's the color of your hat?" "Sorry, I do not know". "B, do you know the color of your hat?" "Sorry, I don't know either." "C, how about you?" "Yes, I know! It is red!"

Is C lying to ease D? If not, how does he figure it out? And why does a seeming uninformative message change the result?

Sure, C is not (as a Bocconi Professor). Here is C's logical process.

If both B and C's hats are white, then A can answer with confidence that his hat was red. But he can not tell, which reveals to B and C that at least one of them is wearing a red hat.

C watches B also admit that he cannot tell his hat color, and then reasons as follows: "If my hat had been white, then B would have answered that he was wearing a red hat, since we both know at least one of us is wearing a red hat. But B could not answer. Therefore, I must be wearing a red hat".

The story is surprising because aside from the apparently innocuous remark of David, the other professors appear to learn from nothing but their own ignorance. Indeed, this is precisely the case. Why does a seeming uninformative message change the result? The fact that there was at least one red hat in the room was mutual knowledge, but not common knowledge, at the first beginning. After D made the announcement, it became common knowledge, B and C could draw inference from A's answer, eventually enabling C to deduce his hat color.

The Issue of Realism

Game Theory is about rational choice of strategies. Rational decision making may require complex calculations and a sophisticated understanding of the other players' motivations. Standard game theory assumes that the players are sophisticated and that they can handle whatever difficult calculations are needed for payoff maximisation. But, in reality, behaviour sometimes diverges from this rationality ideal and it does so in systematic ways. Sometimes real players do not have a shared understanding of the game they are playing. Sometimes a cognitive bias warps how they develop beliefs about each other. Sometimes real players are unable to perform all of the complicated calculations that would be required to determine an optimal strategy. Sometimes players do not fully understand each other's rationality or lack thereof.

We do not expect the outcome of our theoretical models to be perfect indicators of the real world. The relevant question is how closely the theory approximates reality. Where the approximation is poor, there might be an opportunity for the theory, in a normative sense, to play an instructive role in helping people to better understand the strategic settings they face (and to select better strategies). On a descriptive sense, there may be a need to enrich the theory to incorporate real bounds on rationality.

Overview of Solution Concepts

Game Theory is the analysis of rational behaviour in games. This endeavour separates roughly into two components:

- Solution concepts which precisely define concepts of rational behaviour and leads to theories of how games are or should be played.
- Applying the concepts to study specific examples and applications.

Topic 7 focuses on a definition of rationality that operates on the normal form of a game. A player is thought to be rational if:

- through some cognitive process, he forms a belief about the strategies of the others, and
- given this belief, he selects a strategy to maximise his expected payoff

The **Rationalizability** solution concept combines this definition of rationality with the assumption that the game and the players' rationality are common knowledge/belief.

The **Nash Equilibrium** solution concept adds the assumption that a social institution helps to coordinate the players so that their beliefs and actual behavior are consistent.

Topic 8 presents refinements of the rationality concept that are sensitive to individual information sets in the extensive-form representation of the game: **backward induction**, **subgame-perfect equilibrium**, and a variant for games with a negotiation component.

Topic 9 expands the analysis to look at games with exogenous random events (moves of nature), where the solution concepts are called **Bayesian Rationalizability**, **Bayesian Nash Equilibrium** and **Perfect Bayesian Equilibrium**.

Dominance and Best Response

Topic 7-1 covers the two most basic concepts in the theory of strategic interaction: **dominance** and **best response**.

Dominance

Suppose that you are player 1 of the game. Select your own strategy. Strategy U has an interesting property. Regardless of player 2's choice, U gives you a strictly higher payoff than does D. If player 2 plays L, then you obtain 2 from playing U and 1 from playing D. If player 2 selects R, then you obtain 5 by playing U and 4 by playing D. We say that strategy D is **dominated** by strategy U. D should never be played by a rational player 1.

	Player 2	
	L	R
Player 1: U	2, 3	5, 0
D	1, 0	4, 3

1 \ 2	L	C	R
U	8, 3	0, 4	4, 4
M	4, 2	1, 5	5, 3
D	3, 7	0, 1	2, 0

Suppose that you are player 1 of the game. Is any strategy dominated by another strategy? D is dominated by M.

1 \ 2	L	R
U	4, 1	0, 2
M	0, 0	4, 0
D	1, 3	1, 2

Suppose that you are player 1 of the game.
 Is any strategy dominated by another strategy? No pure strategy is dominated by another pure strategy.
 However, a mixed strategy, $(\frac{1}{2}, \frac{1}{2}, 0)$, dominates D. If L, then you obtain 2 from $(\frac{1}{2}, \frac{1}{2}, 0)$ and 1 from D. If R, then you obtain 2 from $(\frac{1}{2}, \frac{1}{2}, 0)$ and 1 from D.

Definition: Dominance
 A pure strategy s_i of player i is dominated if there is a strategy (Pure or mixed) $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all strategy profiles $s_{-i} \in S_{-i}$ of the other players.

Remarks: Strict dominance vs Weak Dominance ($>$ vs $\geq$).
 Playing a dominated strategy is not rational.

The First Strategic Tension

Before enriching the theory, let's use the concept of dominance to identify rational play in a simple application. For both players, strategy C is dominated by strategy D. We would therefore predict that neither player would select C. However, both players would be better off if they each selected C.

1 \ 2	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoners' Dilemma

Group Interests vs. Individual Interests
 Players are jointly better off if they each select C rather than D. However, each has the **individual** incentive to defect by choosing D. Strong individual incentives can lead up to group loss.
Remark 1: What if players can retribute each other? Then it should be explicitly modelled in the game. Changing the payoffs or adding the retribution stage.
Remark 2: What if players have the opportunity to form legally binding contracts? Then it should be included in the game. Theorists do in fact pursue this angle; see Chapter 13 in Watson's Book.
Remark 3: What if players play this game multiple times? Then it should be included in the game. Theorists do in fact pursue this angle; we will pick it up in Topic 8.

Pareto Efficiency

Definition: Pareto Dominance
 A strategy profile s Pareto dominates strategy profile s' if: $u_i(s) \geq u_i(s')$ for all player i and the inequality is strict for at least one player.

Definition: Pareto Efficiency
 A strategy profile s is Pareto efficient if there is no other strategy profile s' that Pareto dominates s .

1 \ 2	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoners' Dilemma

What are the Pareto efficient strategy profiles in Prisoners' Dilemma?
 (C,C), (C,D) and (D,D).
 (D,D) is Pareto dominated by (C,C).

Best Response

Dominance is a good descriptive and prescriptive concept. But it is just the beginning of our development of a theory of behaviour. Indeed, in most games, players have more than one undominated strategy. We move on to explore the process by which players actually select their strategies, at least among those that are not dominated.
 Rational players think about the actions that the other players might take; that is, people form beliefs about one another's behaviour. To maximise the payoff that you expect to obtain — which we assume is the mark of rational behaviour — you should select the strategy that yields the greatest expected payoff given your belief. Such a strategy is called **Best Response** (or Best Reply).

Definition: Best Response
 Suppose player i has a belief $\theta_{-i} \in \Delta S_{-i}$ about the strategy played by the other players. Player i 's strategy $s_i \in S_i$ is a best response if $u_i(s_i, \theta_{-i}) \geq u_i(s'_i, \theta_{-i})$ for every $s'_i \in S_i$.

For any belief θ_{-i} of player i , we denote the set of best responses by $BR_i(\theta_{-i})$.

Example

2 1	L	C	R
U	2, 6	0, 4	4, 4
M	3, 3	0, 0	1, 5
D	1, 1	3, 5	2, 3

Suppose you are P1 with belief $\theta_{-1} = (1, 0, 0)$, what is $BR_1(\theta_{-1})$? $u_1(U, \theta_{-1}) = 2$, $u_1(M, \theta_{-1}) = 3$, $u_1(D, \theta_{-1}) = 1$. $BR_1(\theta_{-1}) = \{M\}$.

Suppose you are P1 with belief $\theta_{-1} = (1/3, 1/2, 1/6)$, what is $BR_1(\theta_{-1})$? $u_1(U, \theta_{-1}) = \frac{1}{3} \times 2 + \frac{1}{2} \times 0 + \frac{1}{6} \times 4 = \frac{8}{6}$, $u_1(M, \theta_{-1}) = \frac{3}{2}$, $u_1(D, \theta_{-1}) = \frac{13}{6}$.
 $BR_1(\theta_{-1}) = \{D\}$.

Remark: Playing a best response to one's belief about the moves of others is not in itself a strategic act. One can think of it as merely a computational exercise associated with rationality. **Forming one's belief is the more important component of strategy.**

Suppose you are to play the following game as P1 and discuss your strategy with your friend, the panda guy. On the basis of your knowledge of P2, you think that P2 will certainly choose strategy A in the game. Therefore you plan to select A as well. Upset with your statement, the panda guy criticises you for "giving in". You should demand the best outcome for yourself by playing B, he asserts.

2 1	A	B
A	1, 3	0, 0
B	0, 0	3, 1

Should you follow the advice of your friend? In fact, you should not. Your actual choice in the game has no direct effect on the other player's thought process or action. Players select their strategies simultaneously and independently. You cannot directly influence how the other player will act in a game such as this one. **The best you can do is to form a well-reasoned opinion about what strategy he will take and then play a best response to this belief.** The rules of the game do not allow you to signal your intentions to the other player by your choice of strategy. So, perform whatever mental gymnastics are necessary for you to settle on a belief about P2's behaviour and then play a best response.

The most substantive component of behaviour is the formation of beliefs. Indeed, herein lies the real science of game theory. Success in games often hinges on whether you understand your opponent better than he understands you. In fact, beliefs are subject to scientific study as well. The study of beliefs of players' is at the center of the study of strategic behaviour.

Dominance and Best Response

There is a precise relation between dominance and best response, the latter of which underpins the theories of behaviour to come. To state relationship, we first define two concepts. For a given game, let UD_i be the set of strategies for player i that are not strictly dominated. Let B_i be the set of strategies for player i that are best responses, over all of the possible beliefs of player i . Mathematically, $B_i = \{s_i \mid \exists \theta_{-i} \in \Delta S_{-i} \text{ s.t. } s_i \in BR_i(\theta_{-i})\}$. If a strategy s_i is a best response to some possible belief of player i , then s_i is contained in B_i .

Example

1 \ 2	(p)	(1-p)
	L	R
U	6, 3	0, 1
M	2, 1	5, 0
D	3, 2	3, 1

Find B_2 and UD_2 . $UD_2 = \{L\}$, $B_2 = \{L\}$.

Find B_1 and UD_1 . $UD_1 = \{U, M\}$, $B_1 = \{U, M\}$. * D is dominated by $(\frac{1}{3}, \frac{2}{3}, 0)$, $UD_1 = \{U, M\}$, p. 57 of Watson for details.
 $B_i = UD_i$?

Proposition: In a finite two-player game, $B_1 = UD_1$ and $B_2 = UD_2$.

A game is finite if each player has a finite number of options, the number of players is finite.

For two-player matrix games, where $B_i = UD_i$, you can most easily compute this set by using the following procedure.

- Look for strategies that are best responses to the simplest beliefs — those beliefs that put all probability on just one of the other player's strategies. These best responses are obviously in the set B_i so they are also in UD_i .
- Look for strategies that are dominated by other pure strategies; these dominated strategies are not in UD_i and thus they are also not in B_i .
- Test each remaining strategy to see if it is dominated by a mixed strategy. This final step is the most difficult, but if you are lucky, then you will rarely have to perform it.

Example

		2	(3/5)	(2/5)
		1	L	R
		W	2, 4	2, 5
	(p)	X	2, 0	7, 1
	(1 - p)	Y	6, 5	1, 2
		Z	5, 6	3, 0

$X \in B_1$, $Y \in B_1$.

$W \notin B_1$, dominated by Z

Check Z. Suppose Z is dominated by $pX + (1-p)Y$. Then, $2p + 6(1-p) > 5$, $7p + 1(1-p) > 3$. $p < 1/4$, $p > 1/3$.

Rationalizability

Topic 7-2 (Chapter 7 in Watson) discusses the implications of combining two fundamental assumptions:

- Players best respond to beliefs (rationality)
- Rationality is common knowledge/belief between the players.

Example

1 \ 2	X	Y	Z
A	3, 3	0, 5	0, 4
B	0, 0	3, 1	1, 2

Suppose that you are player 1 in the following game. What is UD_1 , your set of undominated strategies? $UD_1 = \{A, B\}$. Are we finished with the analysis? Not by a long shot. Put yourself in player 2's shoes. X is dominated by Y. By the rationality of P2, she will not play X. Since you believe she is rational, you should assign zero probability to her strategy X.

1 \ 2	X	Y	Z
A	3, 3	0, 5	0, 4
B	0, 0	3, 1	1, 2

(a)

You might as well strike the column of the matrix corresponding to strategy X.

Striking X leads to a new "game" in which you choose between A and B and your opponent chooses between Y and Z.

In this reduced game, your strategy A is strictly dominated by B. That is, knowing that your opponent will not play X, your only rational strategy is to play B.

1 \ 2	Y	Z
A	3, 3	0, 4
B	3, 1	1, 2

(b)

You can strike strategy A from consideration. You will play B. Interestingly enough, we can take this logic one step further in this game.

Your opponent knows that you know that he will not play X. Thus, recognizing that you are rational, he deduces that you will not play A.

1 \ 2	Z
B	1, 2

(c)

Putting probability 1 on the event that you will play B, his only best response is to play Z.

That is, in the reduced game where you play only B, strategy Y is dominated

Simple logic led to a unique prediction in the example. With rational players and common beliefs/knowledge of rationality, the only outcome that can arise is the strategy profile (B, Z), yielding the payoff vector (1, 2). As in the prisoners' dilemma, our prediction here is a decidedly inefficient outcome. Both players could fare better if the strategy profile (A, X) were played, but neither player finds it in his or her interest to do so.

Rationalizability

1. Delete all of the dominated strategies for each player.
2. Then define R^1 to be the strategy profiles that remain.
3. The next step is to remove any strategies that are dominated in this reduced game and define R^2 as the strategy profiles that remain.
4. Continue this process to identify smaller and smaller sets of strategy profiles, R^3 , R^4 , ..., until no more strategies can be deleted.
5. Let R denote the resulting set of strategy profile — those that survive this iterated dominance procedure. $R^* \downarrow R$

The procedure just illustrated is called iterative removal of (strictly) dominated strategies (or iterated dominance, for short). We can apply the procedure to any normal-form game as follows. First, delete all of the dominated strategies for each player, because a rational player would never use one of these. Then define R^1 to be the strategy profile that remain. Common beliefs/knowledge of rationality therefore implies that it is common knowledge that a strategy profile in R^1 will be played. In other words, essentially the players are interacting in a "reduced" game in which R^1 is the set of strategy profiles. The next step is to remove any strategies that are dominated in this reduced game and define R^2 as the strategy profiles that remain. As before, it is then common knowledge that a strategy profile in R^2 will be played. Continue this process to identify smaller and smaller sets of strategy profiles, R^3 , R^4 , ..., until no more strategies can be deleted. Let R denote the resulting set of strategy profiles — those that survive this iterated dominance procedure.

A very simple and precise theory of behaviour is represented here.

- Each player should play a best response to her belief. In this way, the belief rationalises playing the strategy.
- Furthermore, each player should assign positive probability only to strategies of the other players that can be similarly rationalised.
- Combining rationality (best-response behaviour, never playing dominated strategies) with common knowledge **implies, and only implies**, that players will play strategies that survive iterated dominance. We call these the **rationalizable strategies**.

In two-player games, dominance and best response imply the same restriction. Iterated dominance is identical with the procedure in which strategies that are never best responses are removed at each round. The logic of rationalizability depends on common belief/knowledge of rationality and the game.

The Second Strategic Tension

Observe that rationalizability does not always lead to a unique strategy profile. In fact, it often has little bite in games. Rationalizability merely requires that the players' beliefs and behaviour be consistent with common belief of rationality. It does not require that their beliefs are correct. We have thus identified another tension in games, which is called strategic uncertainty. One manifestation of strategic uncertainty is the coordination problem, which players often face in games such as the battle of sexes.

Location and Partnership

Topic 7-2 presents rationalizability and iterated dominance. While now, with topic 7-3 we cover applications of rationalizability: two (Location and 70 Percent) in the lecture, one (Partnership) for self-study.

Location game

We consider a very simple model of location choice by competitors. Suppose two people — call them Chris and Pat (C and P) — work for the same large soft drink company. Their job is to sell cans of the company's soda at a popular beach. They will be working at the same beach and, by company policy, must charge the same price. The company has committed to give each salesperson a commission of 25 cents per can sold. Each salesperson must work from a booth that is stationary on the beach. The only decision that each person has to make is where on the beach to set his booth at the beginning of the day. Chris and Pat select the locations for their booths independently and simultaneously.

The beach is divided into nine regions of equal size,

1	2	3	4	5	6	7	8	9
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Booths may be located in any one of these regions (and it is possible for the salespeople to locate in the same region). On a given day, 40 people in each region will each wish to purchase a soda. Thus, if a salesperson serves all of the customers in a single region, he will earn \$10. Customers walk to the nearest booth to make their purchases. If the customers of a given region are indifferent between the two, half go to one and half go to the other. C and P seek to maximise their income.

We first represent the game in normal form. $N = \{C, P\}$; $S_i = \{1, 2, \dots, 9\}$, $i = C, P$; $u_i(s_C, s_P) = ? \Rightarrow u_C(2, 5) = 3 \cdot 10 = 30$, $u_P(2, 5) = 6 \cdot 10 = 60$. $u_C(1, 9) = 45 = u_P(4, 9)$

We begin our analysis by evaluating strategy 1 and 2 against all of the opponent's strategies.

1	2	3	4	5	6	7	8	9
---	---	---	---	---	---	---	---	---

$$1) S_p = 1, U_c(1,1) = 45, U_c(2,1) = 80.$$

$$U_c(1, S_p) < U_c(2, S_p) \text{ for all } S_p \in S_p$$

$$2) S_p = 2, U_c(1,2) = 10, U_c(2,2) = 45.$$

Strategy 2 strictly dominates strategy 1 for both players.

$$3) S_p = 3, U_c(1,3) = 15, U_c(2,3) = 20$$

Strategy 8 strictly dominates strategy 9 for both players.

In fact, strategies 1 and 9 are the only dominated strategies in the game. Why? 2 is a best response to the opponent selecting 1; 3 is a best response to the opponent selecting 2...

$$R_1^1 = \{2, 3, 4, 5, 6, 7, 8\}, R_1^2 = \{3, 4, 5, 6, 7\}, R_1^3 = \{4, 5, 6\}, R_1^4 = \{5\}, R_1^5 = \{5\}, R = \{(5, 5)\}.$$

Applications of The Location Model

Rationalizability implies that the players, in their quest for a large share of the market, both locate in the same region at the center of the beach. The model solidifies some intuition that can be applied more generally.

- Location of Gas Station.
- Product variety: taste for sweet cereal.
- Politics: politicians, just like salesmen with consumers, seek to capture the majority of voters.

Partnership (See Chapter 8 of Watson Book)

A Partnership game

Firm's Profit: $x, y \in [0, 4]; P = 4(x+y+cx y)$ with $c \in [0, 1/4]$. x^2, y^2 ; $U_1 = 2(x+y+cx y) - x^2$ and $U_2 = 2(x+y+cx y) - y^2$

① Suppose you believe $y=1$, $U_1(x, 1) = 2(x+1+cx) - x^2$ and $\frac{\partial U_1}{\partial x} = 2+2c-2x = 0 \Rightarrow x = 1+c$

② Suppose you are P1 and have a belief with expectation $\bar{y}$. Then expected payoff is $2(x+\bar{y}+cx\bar{y}) - x^2$. F.O.C.: $2+2c\bar{y}-2x = 0 \Rightarrow x^* = 1+c\bar{y}$.

Then $BR_1(\bar{y}) = 1+c\bar{y}$ and $BR_2(\bar{x}) = 1+c\bar{x}$ with $c \in [0, 1/4]$, $x, y \in [0, 4]$ and $R_i^0 = [0, 4]$, then $R_1^1 = [1+c(0), 1+c(4)]$ and $R_2^1 = [1+c(4), 1+c(1+4c)]$.

From which we can see that $R_i^k = [a^k, b^k]$ where $a^{k+1} = 1+c \cdot a^k$ and $a^0 = 0$ while $b^{k+1} = 1+c \cdot b^k$ with $b^0 = 4$. Hence $a^k \uparrow a$ and $b^k \downarrow b$ and we have that $a = 1+c \cdot a \Rightarrow a = \frac{1}{1-c}$

$R = \left\{ \frac{1}{1-c} \right\}^2$ and $P = \frac{8-4c}{(1-c)^2}$ and $U_i = \frac{3-2c}{(1-c)^2}$. Efficient Outcome: $\max_{x,y} U_1 + U_2$ $\hat{x} = \hat{y} = \frac{2}{1-2c} > \frac{1}{1-c}$

Battle of The Sexes

A group of three students will be in a breakout room. First two in the BBoard are P1 and P2 respectively. 3rd is moderator. P1 and P2 play BoS by submitting strategies to moderator via private chat.

1		2	
		A	B
A	A	1, 3	0, 0
	B	0, 0	3, 1

Moderator announces the outcome.

All groups play the BoS game once. **No public messages other than announcing game outcome.** Then...

- Group 1,4,7 play the game (step 3,4) five more times.
- Group 2,5,8 play the game (step 3,4) twice more: chat between P1 and P2 allowed.
- Group 3,6,9 play the game (step 3,4) twice more: moderator can make suggestions.

Topic 7-2 (Chapter 7 in Watson) discusses the implications of combining two fundamental assumptions:

- Players best response to beliefs (rationality);
- Rationality is common knowledge/belief between the players.

The concept is quite weak in that no additional assumptions about behaviour are made. We do not assume that each player's beliefs are consistent with the strategies actually used by the other players.

There are many settings in which rationalizability is the appropriate behavioural concept (buying a house, locating a friend on your first trip to Disneyland, interview at a new company, military conflicts).

In other settings, strategic uncertainty is resolved through a variety of social institutions (repeated social play with norms, pre-play communication between the players, an outsider mediator suggests strategies).

The point of Topic 8 (Chapter 9, 10 in Watson) is to begin to explore rational behaviour when actions and beliefs are coordinated by social institutions. The underlying idea is that through some special force (such as those just discussed) behaviour in a game is coordinated, or congruous. Congruity can refer to consistent and regular behaviour in a game that is played over and over in society or by the same parties who interact repeatedly. Congruity can also refer to behaviour in a one-shot game in which communication or social history has aligned the beliefs of each player with the strategies of the others. We shall examine the common element — **the combination of best-response behaviour and reduced strategic uncertainty.**

Nash Equilibrium

The simplest notion of congruity is that the players are best responding in a setting of strategic certainty. In other words, the players coordinate on a single strategy profile. In such a case, the players' belief and behaviour is consistent, with each player's belief about another player's strategy concentrated on the actual strategy that the other player uses. Because of these accurate beliefs, the players are best responding to each others' strategies. To say it differently, the players' strategies are "mutual best responses".

Definition: Nash Equilibrium

A strategy profile $s \in S$ is a (pure strategy) Nash Equilibrium if and only if $s_i \in BR_i(s_{-i})$ for each player i . That is, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for every $s'_i \in S_i$ and each player i .

Nash Equilibrium: Interpretation

The Nash Equilibrium has found many uses in economics, partly because it can be usefully interpreted in a number of ways.

- **Stable point of adjustment** (Social convention, evolution, biology).
- **Self-enforcing agreement** (In the self interest of each player to follow the agreement if others do. No unilateral deviation).
- **Persuasive advice** (Any advice that was not an equilibrium would have the unsettling property that there would always be some player for whom the advice was bad).

Nash Equilibrium: Examples

1 \ 2	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Matching Pennies

1 \ 2	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoners' Dilemma

1 \ 2	Opera	Movie
Opera	2, 1	0, 0
Movie	0, 0	1, 2

Battle of the Sexes

1 \ 2	H	D
H	0, 0	3, 1
D	1, 3	2, 2

Hawk-Dove/Chicken

1 \ 2	A	B
A	1, 1	0, 0
B	0, 0	1, 1

Coordination

1 \ 2	A	B
A	2, 2	0, 0
B	0, 0	1, 1

Pareto Coordination

1 \ 2	P	D
P	4, 2	2, 3
D	6, -1	0, 0

Pigs

Definition: Strict Nash Equilibrium

A strategy profile $s \in S$ is a strict Nash Equilibrium if and only if $s_i = BR_i(s_{-i})$ for each player i . That is, $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for every $s'_i \in S_i$. And each player i .

Player i 's strategy s_i is the only best response to s_{-i} .

NE in Matrix Games

An algorithm for finding NE in matrix games.

- 1 Find the best responses of player 1 to each of the strategies of player 2.
- 2 Find the best responses of player 2 to each of the strategies of player 1.
- 3 NE = Mutual Best Responses.

1 \ 2	X	Y	Z
J	5, 6	3, 7	0, 4
K	8, 3	3, 1	5, 2
L	7, 5	4, 4	5, 9
M	3, 4	7, 5	3, 3

(a)

1 \ 2	k	l	m
w	8, 6	6, 8	0, 0
y	6, 8	8, 6	0, 0
z	0, 0	0, 0	5, 5

Nash Equilibrium Applications

Cournot Duopoly Model

Firms 1 and 2 produce exactly the same good, say marble brick. Simultaneously and independently, the firms select the number of bricks to produce, $q_1, q_2 \geq 0$. All of the brick is sold, but the price that consumers are willing to pay depends on the total output $q_1 + q_2$. The price is given by the simple function $p = 1000 - q_1 - q_2$. Production cost is 100 euros per brick. Firms wish to maximise their profits.

Step 1. Represent the Cournot Duopoly in a normal form game.

$$N = \{1, 2\}, S_i = [0, +\infty), i = 1, 2. \quad u_1(q_1, q_2) = (1000 - q_1 - q_2)q_1 - 100q_1. \quad u_2(q_1, q_2) = (1000 - q_1 - q_2)q_2 - 100q_2.$$

Step 2. Calculate the best-response functions.

Because we will be looking for a Nash Equilibrium strategy profile, we can think of each firm's best response as a function of the other firm's quantity (rather than a belief about this quantity). u_1 is a downward parabola, as a function of q_1 . $\frac{\partial u_1}{\partial q_1} = 1000 - 2q_1 - q_2 - 100 = 0 \Rightarrow BR_1(q_2) = 450 - q_2/2$.

The game is symmetric, so $BR_2(q_1) = 450 - q_1/2$.

Step 3. Find mutual best responses. Look for q_1^* and q_2^* such that $BR_1(q_2^*) = q_1^*$ and $BR_2(q_1^*) = q_2^* \Rightarrow q_1^* = q_2^* = 300$.

However, (300, 300) is not efficient (225, 225) maximizes the sum of profits and Pareto dominates NE.

Can you find other strategies that Pareto dominate NE?

Bertrand Duopoly Model

From quantity competition to price competition. Suppose that the two firms simultaneously and independently set prices, p_1, p_2 . They are forced to produce exactly the number of bricks demanded by customers at these prices. Total demand is given by $Q = 1000 - p$. Consumers purchase brick from the firm that charges the lower price. Demand is split evenly if prices are equal. Production cost is 100 euros per brick. Firms wish to maximise their profits.

Step 1. Represent the Bertrand Duopoly in a normal form game.

$$N = \{1, 2\}, S_i = [0, +\infty), i = 1, 2. \quad u_1(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 > p_2 \\ (1000 - p_1)p_1 - 100(1000 - p_1) & \text{if } p_1 < p_2 \\ \frac{1}{2}[(1000 - p_1)p_1 - 100(1000 - p_1)] & \text{if } p_1 = p_2 \end{cases} \quad u_2 \text{ behaves equivalently.}$$

Step 2. Find a NE: (p_1, p_2) that p_i is BR to p_j . First, if $\exists p_i < 100$, negative profit (for at least one firm). Therefore, $p_i \geq 100$ for $i = 1, 2$.

Suppose $p_i > 100$ for $i = 1, 2$. Both firms deviate. Suppose $p_i > p_j = 100$ for some i . Firm j can raise profit from 0 by $p_j + \epsilon$.

Suppose $p_i = p_j = 100$. Neither firm can gain by changing price.

Exercise Now, suppose firms have different production costs. Prove that there is no pure strategy NE.

Median Voter Theorem

Two political candidates (P1 & P2) decide where to locate on the political spectrum $[0, 1]$. Voter's ideologies are distributed across the political spectrum with a pdf f on $[0, 1]$. Candidates simultaneously and independently select their policy locations. Each voter is assumed to vote for the candidate who is located closest to the citizen's ideal point.

A candidate wins by obtaining more votes than the other candidate does. A candidate gets a payoff of 1 if she wins, 0 if loses, $1/2$ if ties.

Let us analyze the game by looking for a Nash Equilibrium. First, we claim $s_1 = s_2 = x$ in equilibrium.

- Suppose $s_1 \neq s_2$.
- If P1 wins then P2 can change to s_1 .
- If P2 wins then P1 deviates.
- If tie, both move to the middle.

Now, where is s ? Location game in Topic 7 suggests the "center". $x = 0.5$? No. x^* is the median ideology, $\int_0^{x^*} f(t) dt = 0.5$.

(x^*, x^*) is the UNIQUE NE.

- Any unilateral deviation from (x^*, x^*) would get fewer votes and lose.
- For any other (x, x) , moving towards x^* is profitable.

Mode Game

Each of you can pick one number from $\{1, 2, \dots, 20\}$. The mode x will be calculated. If there are 2+ modes, the highest is chosen.

If $x < 20$, then everyone who selected $x + 1$ get $(x + 1)/10$ bonus points in the final, and others get zero.

If $x = 20$, then everyone who selected 1 get $1/10$ bonus points in the final and others get zero.

Mixed Strategy Nash Equilibrium

Definition: Mixed Strategy Nash Equilibrium

Consider a strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, where $\sigma_i \in \Delta S_i$ for each player i . Profile σ is a mixed-strategy Nash Equilibrium if and only if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for each $s_i' \in S_i$ and each player i .

σ is a mixed-strategy Nash Equilibrium if and only if σ_i is a best response to σ_{-i} for every player i .

For a mixed strategy to be a best response, it must put positive probability only on pure strategies that are best responses.

The game players standing for a large population of agents. Each of the agents chooses a pure strategy, and the payoff depends on the fraction of agents choosing each strategy. The mixed strategy hence represents the distribution of pure strategies chosen by each population.

Existence of NE

Proposition: Existence of Nash Equilibrium

Every finite game (having a finite number of players and a finite strategy space) has at least one Nash Equilibrium in pure or mixed strategies.

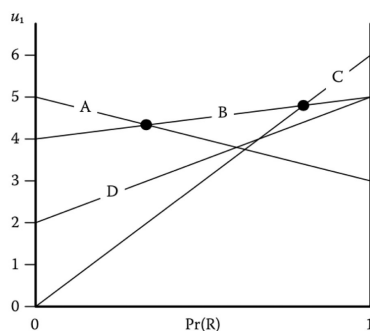
Procedure for Finding Mixed-Strategy NE

Procedure for finding mixed-strategy equilibria:

- Calculate the set of rationalizable strategies by performing the iterated-dominance procedure.
- Restricting attention to rationalizable strategies, write equations for each player to characterise mixing probabilities that make the other player indifferent between the relevant pure strategies.
- Solve these equations to determine equilibrium mixing probabilities.

	L	R
A	5, 0	3, 6
B	4, 4	5, 1
C	0, 5	6, 3
D	2, 5	5, 4

(a) A normal form game



(b) Graph of u_1 vs. ΔA_2

More on Extensive Form

Rationalizability and Nash equilibrium are based on the normal form. Topic 9 focuses on extensive form. Features of extensive form suggest **refined** version of Rationalizability and Nash equilibrium: Information, How to look ahead and evaluate other's responses.

An extensive form game is defined by a tree that has been **properly constructed** and **labelled**, a **directed graph**.

Trees consist of **nodes** that are connected by **branches**. Each branch is an arrow pointing from one node to another.

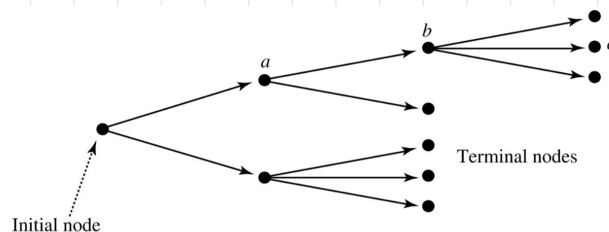
Start from a given node and trace through the tree by following arrows.

- Nodes that can be reached in this manner are called **successors** of the node at which you start.
- Branches from a given node point to its **immediate successors**.
- A node's **predecessors** and **immediate predecessor** are defined similarly by tracing backward through the tree.

Rule 1

A tree starts with the initial node and ends at terminal nodes (from which no arrows extend). We would like the initial node clearly to designate the "beginning" of the tree, and so we impose the following rule on trees:

Tree Rule 1: Every node is a successor of the initial node, and the initial node is the only one with this property.



Rule 2

A path through the tree is a sequence of nodes that:

- Starts with the initial node
- Ends with a terminal node
- Has the property that successive nodes in the sequence are immediate successors of each other.

No two paths should cross.

Tree Rule 2: Each node except the initial node has exactly one immediate predecessor. The initial node has no predecessors.

Rule 3

Remember that branches represent actions that the players can take at decision nodes in the game. We label each branch in a tree with the name of the action that it represents.

Tree Rule 3: Multiple branches extending from the same node have different action labels.

Rule 4 and Rule 5

Information sets represent places where players have to make decisions. An information set is a set of nodes between which a player cannot distinguish when making a decision. The concept of an information set adds the following two rules for trees.

Tree Rule 4: Each information set contains decision nodes for only one of the players.

Tree Rule 5: All nodes in a given information set must have the same number of immediate successors and they must have the same set of action labels on the branches leading to these successors.

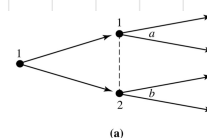


Figure: players do not know who is to make a decision

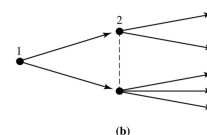


Figure: able to distinguish between nodes in one info set

Perfect Recall

Players remember their own past actions as well as any other events that they have observed.

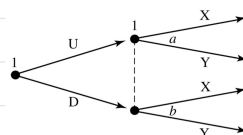


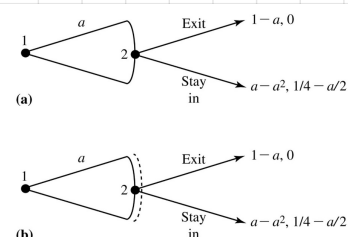
Figure: Imperfect recall

Perfect vs. Imperfect Info

Perfect Information: Every information set of the extensive form is a single node (No dashed line in the tree).

Imperfect Information: Otherwise.

What are the strategy spaces of player 2 in the following games?



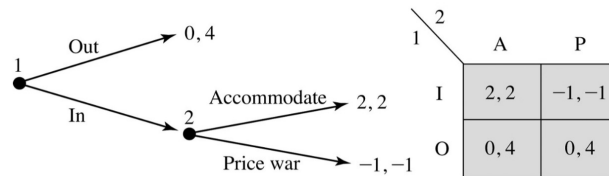
Sequential Rationality and Subgame Perfection

Example

There are two firms in this market game: a potential entrant (player 1) and an incumbent (player 2). The entrant decides whether or not to enter the incumbent's industry.

- If the entrant stays out, then the incumbent gets a large profit and the entrant gets zero.
- If the potential entrant does enter the industry, then the incumbent must choose whether or not to engage in a price war.

If the firm triggers a price war, then both firms suffer. If the incumbent accommodates the entrant, then they both obtain modest profits.



There are two NE (I, A) and (O, P). Take a closer look at (O, P).

- The entrant prefers not to enter if she expects the incumbent to initiate a price war.
- The incumbent's thought process here consists of an **ex ante** evaluation; that is, the incumbent is willing to **plan** on playing the price-war strategy when he believes that his decision node will not be reached.

Is player 2's threat to initiate a price war incredible in reality? If not, how should P1 respond? To summary, **an equilibrium specifies an action that is irrational conditional on a particular information set being reached.**

Sequential Rationality

More should be assumed about players than that they select best responses **ex ante** (from the point of view of the beginning of the game). Players ought to demonstrate rationality whenever they are called on to make decisions. This is called sequential rationality.

Definition: Sequential Rationality

An optimal strategy for a player should maximize his or her expected payoff, conditional on every information set at which this player has the move. That is, player i 's strategy should specify an optimal action from each of player i 's information sets, even those that player i does not believe (ex ante) will be reached in the game.

If sequential rationality is common belief between the players (at every information set), then each player will "look ahead" to consider what players will do in the future in response to her move at a particular information set.

To operationalise the notion of sequential rationality, one has to deal with some intricacies regarding limitations on the beliefs players have about each other at different information sets.

Backward induction, subgame perfect Nash equilibrium, forward induction.

Backward Induction

Identify an optimal action for each information set by working backward in the game tree. Consider a game of perfect information (there are no dashed lines in the extensive-form diagram).

- One can start by looking at the decision nodes that are immediate predecessors of only terminal nodes. There is no need to think about the behavior of other players.
- We assume that he will select the payoff-maximising action. Let us call the other actions "dominated".
- The procedure then moves to evaluate decision nodes whose immediate successors are either terminal nodes or the nodes we already evaluated. From each node in this second class, the payoff consequences of each action are clear because the player on the move can anticipate how other players will behave later.
- The process continues all the way back to the initial node.

Definition: Backward Induction

This is the process of analyzing a game from the end to the beginning. At each decision node, one strikes from consideration any actions that are dominated, given the terminal nodes that can be reached through the play of the actions identified at successor nodes.

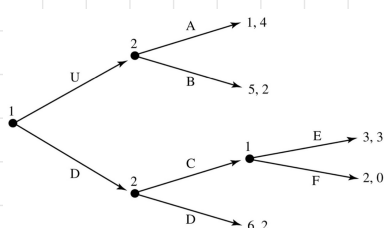


Figure: Backward induction

The backward induction procedure identifies a unique strategy profile for every game of perfect information that has no "payoff ties" (where a player would get the same payoff at multiple terminal nodes). The procedure is also well defined and meaningful for perfect-information games with payoff ties, but in this case it can lead to more than one strategy profile. Further, it can be applied even to some games with nontrivial information sets but it is dodgy in general.

Result

Every finite game with perfect information has a pure-strategy NE. Backward induction identifies an equilibrium.

Sticking with games of perfect information, regardless of payoffs, the backward induction procedure is well- defined.

Subgame Perfection

Another direction to represent sequential rationality:

- Start with the concept of Nash Equilibrium and then think of **refinements** based on the consideration of sequential rationality. For any given game, we can calculate all of the Nash Equilibria and then remove the ones that violate sequential rationality.

What is "violate sequential rationality"? NE + Subgame.

Definition: Subgame

Given an extensive-form game, a node x in the tree is said to initiate a subgame if neither x nor any of its successors are in an information set that contains nodes that are not successors of x . A subgame in the tree structure defined by such a node x and its successors.

We can analyze a subgame as a game in its own right.

How to find a subgame?

- Take any node x in an extensive form. Examine the collection of nodes given by x and its successors.
- If there is a node y that is not a successor of x but is connected to x or one of its successors by a dashed line, then x does not initiate a subgame.

How many subgames in the right figure? Two: the entire game and the subgame starting at node y .

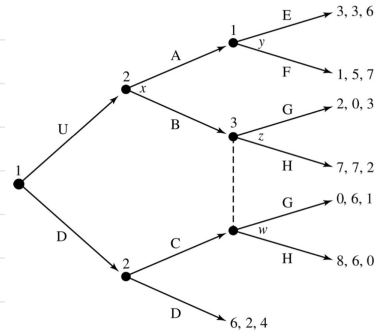


Figure: Subgames

Once players are inside a subgame, it is common knowledge between them that they are inside it.

Subgames are self-contained extensive forms - meaningful trees on their own.

Subgames that start from nodes other than the initial node are called **proper subgames**.

Observe that in a game of perfect information, every node initiates a subgame.

Definition: Subgame perfect Nash Equilibrium (SPNE)

A strategy profile is called a subgame perfect Nash equilibrium if it specifies a Nash equilibrium in every subgame of the original game.

A solution concept should be consistent with its own application from anywhere in the game where it can be applied. Because Nash equilibrium can be applied to all well-defined extensive-form games, subgame perfect Nash equilibrium requires the Nash condition to hold on all subgames.

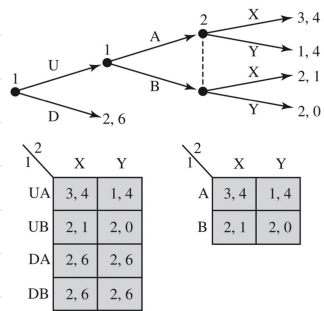


Figure: Subgame perfection

How many subgames? 2.

How many NE? (UA, X), (DA, Y), (DB, Y).

How many SNE? The only SNE is (UA, X), why?

The proper subgame has one NE (A, X).

(DA, Y) and (DB, Y) are not subgame perfect, because they do not specify Nash equilibria in the proper subgame.

The subgame perfection concept requires equilibrium at **every subgame**, meaning that if any particular subgame is reached, then we can expect the players to follow through with the prescription of the strategy.

This is an important consideration, **whether or not a given equilibrium actually reaches the subgame** in question. Nash equilibrium (DA, Y) entails a kind of threat that the players jointly make to give player 1 the incentive to select D at his first information set. But the threat is not legitimate.

In the proper subgame, (A, Y) is about as stable as potassium in water (not very stable :)). Player 1 has the strict incentive to deviate from this prescription.

SPE is a refinement of NE: An SPE is a NE. Such a profile must specify a Nash equilibrium in every subgame, one of which is the entire game. For games of perfect information, backward induction yields subgame perfect equilibria.

How to calculate SNE:

- Direct approach (harder): Examining the matrices corresponding to all of the subgames and locating Nash equilibria.
- Backward-induction procedure: consider smallest subgames, then "prune" the tree.

Stackelberg Duopoly

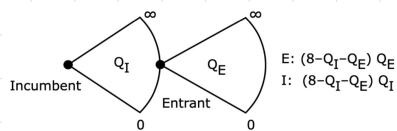


Figure: Stackelberg Duopoly

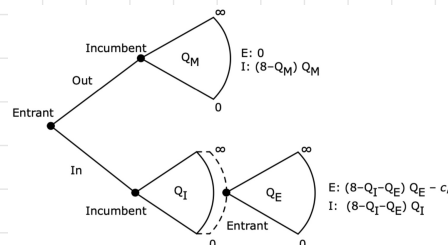


Figure: Stackelberg Duopoly with Entry

What is SPE? 1. Each Q_I initiates a subgame for P_E
 (Backward Induction) 2. $\max_{Q_E} (8 - Q_I - Q_E) Q_E \Rightarrow Q_E^* = BR(Q_I) \Rightarrow \frac{\partial (8 - Q_I - Q_E) Q_E}{\partial Q_E} = 0 \Rightarrow Q_E^* = \frac{8 - Q_I}{2} = \dots$
 3. $\max_{Q_I} (8 - Q_I - Q_I) Q_I \Rightarrow \max_{Q_I} (8 - \frac{(8 - Q_I)}{2} - Q_I) Q_I \Rightarrow Q_I^* = \dots$

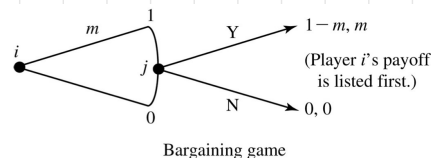
Definition: Continuation Payoff

For a given node x in a game, a player's **continuation payoff** is the payoff that this player will eventually get contingent on the path of play passing through node x .

Bargaining

Ultimatum Game

Two pirates, A and B, find a pile of gold in treasure island. A and B have to decide how to split the treasure. Their friend and consultant, C, proposes the following method. A proposes a way to split, $(1 - m, m)$. B decides to accept A's offer or not. If B accepts, then A and B will get $1 - m$ and m portion of the treasure respectively. If B declines, C will throw the gold to the sea so that neither of them gets anything.



How many subgames does this game have? Infinite. Every decision node for player j initiates a subgame.

Claim 1: Player j has only two sequentially rational strategies: (s_j^*) accept all offers; $(\hat{s}_j)$ accept all offers of $m > 0$ and reject $m = 0$.

- Consider the subgame following offer $m > 0$: reject leads 0, accept leads $m > 0 \rightarrow$ Accept.
- Consider the subgame following offer $m = 0$: reject leads 0, accept leads $m = 0 \rightarrow$ Indifferent.

Any SPE must include (s_j^*) or $(\hat{s}_j)$ on the part of player j .

Claim 2: $m = 0$ and (s_j^*) is a SPE.

Claim 3: $(\hat{s}_j)$ **cannot** be part of an equilibrium. Player i has **no** well-defined best response to $\hat{s}_j$.

There is a unique SPE of the ultimatum game: player i selects $m = 0$, and player j selects s_j^* . The payoff is $(1, 0)$. The equilibrium is efficient. The player who is in the position of making a take-it-or-leave-it offer has great bargaining power.

Two-Period, Alternating-Offer game

Multiple offers and counteroffers by the parties over time: house sell, market. Offer and counteroffers take time. In home sales, one party may wait a week or more for an offer to be considered and a counteroffer returned. Most people discount the future relative to the present. To incorporate discounting into game-theoretic models, we use the notion of **discount factor**.

A discount factor δ_i for player i is a number used to deflate a payoff received next period so that it can be compared with a payoff received now. Suppose a period is 1 week.

Question 1: Do you prefer receiving x euros today or next week? Most people would prefer today.

- People are generally impatient.
- Opportunity to spend money earlier rather than later.
- Earn interest.

Question 2: What is the amount of y euros receiving today that is indifferent with x euros receiving next period?

- $y = \delta_i x$
- $\delta_i \in (0, 1)$
- Larger δ_i correspond to greater patience.
- Payday lenders usually charge interest of 15 - 20 for every 100 borrowed, APR 400%

C proposes a new two-period, alternating-offer bargaining game to A and B. In the first period, player 1 makes an offer m_1 . After observing the offer, player 2 decides whether to accept or reject it. If accepted, the game ends with P1 receiving $1 - m_1$ and P2 obtaining m_1 . If rejected, then in the second period the roles reversed. Player 2 makes a counteroffer m_2 , which player 1 either accepts or rejects. If P1 accepts P2's offer, then P1 gets $\delta_1 m_2$ and player 2 gets $\delta_2 (1 - m_2)$. If player 1 rejects player 2's offer, then both players obtain 0.

Observe that the two-period, alternating-offer game consists basically of repetition of the ultimatum-game structure. In the first period, the players interact as in the ultimatum game, with player 1 playing the role of i and player 2 the role of j . If the offer is rejected, then time passes into the second period, where players interact in an actual ultimatum game with the roles reversed. The game ends after two periods. If an agreement is reached in the second period, then the payoffs are discounted relative to the first period.

If player 2 rejects m_1 , then he effectively induces play of the ultimatum game in the following period. Then subgame perfect implies that P1 accepts all counteroffers and P2 chooses $m_2 = 0$. From the perspective of period 1, these continuation values of players are $\delta_1 \times 0$ and $\delta_2 \times 1$.

If P2 rejects m_1 , he will obtain the full surplus in the next period, which is worth δ_2 in the first period. Therefore, P2 must accept any $m^1 > \delta_2$, reject any $m^1 < \delta_2$, and be indifferent when $m^1 = \delta_2$. Only one thing has yet to be determined regarding behavior following P1's initial offer: whether P2 accepts or rejects the offer $m^1 = \delta_2$.

As with the ultimatum game, the only NE has P2 accepting $m^1 = \delta_2$; furthermore, P1 makes this offer. Thus, there is a unique SPE of the two-period game. $m^1 = \delta_2$.

- P1 offers $m^1 = \delta_2$
- P2 accepts if and only if $m^1 \geq \delta_2$
- If $m^1 < \delta_2$, P2 always offer $m_2 = 0$.
- P1 accepts any m_2 .

The equilibrium yields a payoff of $1 - \delta_2$ to player 1 and δ_2 to player 2.

Patience is positively related to bargaining power.

- If P2 is impatient, then her equilibrium payoff is close to 0.
- The relation between patience and bargaining power is supported more generally, for example, a T-period game.

The equilibrium entails agreement made in the first period.

- The equilibrium of the general T-period game also features agreement without delay.
- Although the bargaining games studied thus far provide a theory of bargaining power, they do not yield a theory of bargaining delay or inefficiency.
- However, delay and inefficiency are regular occurrences in real bargaining settings.
- People do not always have complete information about each other in the real world?

Infinite-Period, Alternating-Offers Game

Let's enrich our model further. C proposes a newer version of the game that runs for a potentially infinite number of periods. There is no final period in which bargaining takes place. It looks like more complex than a T-period game, however searching for a SPE is not harder. A special property (stationarity): the subgames starting from any period t are exactly the same as the subgames starting from period $t + 2$.

SPE specifies an equilibrium on every subgame. Let us look for an equilibrium of the infinite-period game that is **stationary** in the sense that **a player makes the same offer whenever he is on the move**.

Also suppose that these offers are accepted. Suppose P1 offers m_2 in odd periods and P2 offers m_1 in even periods. The continuation payoff vector from any odd period is $(1 - m_2, m_2)$. From the perspective of the current period, not discounted to the beginning of the game. The continuation payoff vector from any even period is $(m_1, 1 - m_1)$.

Player i 's equilibrium offer puts player j in the position of being indifferent between accepting the offer and rejecting it in favour of going on to the next period. Accepting gives m_j in the current period; rejecting leads to $1 - m_i$ in the next period, that is $\delta_j(1 - m_i)$ discounted to current period.

$$w_i = \delta_j(1 - w_j)$$

That is $w_2 = \delta_1(1 - w_1)$ and $w_1 = \delta_2(1 - w_2)$.

$$w_1 = \frac{\delta_1(1 - \delta_2)}{1 - \delta_1\delta_2} \quad \text{and} \quad w_2 = \frac{\delta_2(1 - \delta_1)}{1 - \delta_1\delta_2}$$

One can prove that the outcome sketched is the unique subgame perfect equilibrium outcome in the infinite-period game. The equilibrium features agreement in the first period and yields payoff vector:

$$\left(\frac{1 - \delta_2}{1 - \delta_1\delta_2}, \frac{\delta_2(1 - \delta_1)}{1 - \delta_1\delta_2} \right)$$

i 's payoff increases as δ_i increases and δ_j decreases.

Suppose $\delta_1 = \delta_2 = \delta$. Then payoff vector converges to $(0.5, 0.5)$ as $\delta \rightarrow 1$.

Another Pirate Game

There are five pirates (in strict order of seniority A, B, C, D and E) who found 100 gold coins. They must decide how to distribute them. The pirate world's rules say that the most senior pirate first proposes a plan. The pirates, including the proposer, then vote on whether to accept. If the majority accepts the plan, the coins are dispersed and the game ends. In case of a tie vote, the proposer has the casting vote. If the majority rejects the plan, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to being the system again. The process repeats until a plan is accepted or if there is one pirate left.

Pirates have lexicographic preferences:

- Each pirate wants to survive.
- Given survival, each pirate wants to maximise the number of gold coins he receives.
- Each pirate would prefer to throw another overboard, if all other results would otherwise be equal.

They are rational and have common belief in rationality. The pirates do not trust each other, and will neither make nor honour any promises between pirates.

Incomplete Information

Model Private Information

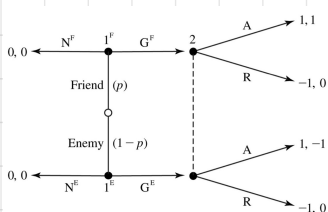
How to model private information?

- Incorporate random events in a game that are out of players' control.
- **Move of Nature**
 - Nature is a player in the game, whom we may call player 0
 - But nature is a non strategic player.
 - Nature's actions are defined by a fixed probability distribution and not by incentives.
- Texas hold 'em example.
 - Players' hold cards are selected by nature at the start of the game.
 - Each player immediately observes her hold cards;
 - Each player does not know other players' cards.

Incomplete Information

Incomplete information refers to games having moves of nature that generate asymmetric information between the players. Theorists use the term **type** to indicate the different moves of nature that a single player privately observes. Often, a player's private information has to do with his or her own personal attributes and tastes, to which the term type is particularly well suited. For example, we can speak of the different types of home buyer distinguished by different valuations of the house.

The Gift Game



P1 has private information on nature's move.

- P1's type determines P2's value of accepting a gift.
- Private info often concerns a player's own payoffs, it sometimes has to do with the payoffs of others.

Rational play requires P1 who knows his own type to think about what he **would have done** had he been **another** type.

- Suppose P1 is a friend and would like to send a gift.
- P2 does not P1's type.
- P1's optimal decision depends on how will P2 react.
- P2's optimal decision depends not only on friend P1 but also on enemy P1.
- Put oneself to the other player's shoes.

Figure: The gift game

Bayesian Normal Form

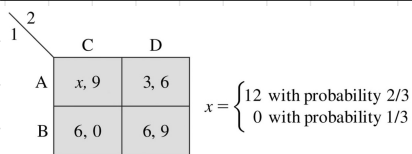
Bayesian normal form: Normal-form version of a game with nature. Same procedure as in previous topics... With a bit more work to compute payoffs by averaging over the random events in the game.

For example, $(G^F G^E, A)$:

- $U_1(G^F G^E, A) = p \cdot 1 + (1-p) \cdot 1 = 1$
- $U_2(G^F G^E, A) = p \cdot 1 + (1-p) \cdot -1 = 2p - 1$

A game with Incomplete Information

Bayesian Nash Equilibrium and Rationalizability



This is not the true normal form of the game because player 1 observes x before making his decision.

Figure: Another game with incomplete information

There are two methods of evaluating Bayesian games.

- Compute rationalizability and Nash Equilibrium on the Bayesian normal form.
- Nash equilibrium and rationalizability concepts can be applied directly to any game with random events.
- Bayesian Nash equilibrium and Bayesian rationalizability.

The second method entails treating the types of each player as separate players.

- For example, if player 1 is one of two types, A and B, then it may be helpful to think of 1A and 1B as distinct players.
- Treating types separately often simplifies the analysis of infinite games, where calculus may be required.

Next we present two examples, one for each method.

		A	R
G ^F G ^E		1, 2p - 1	-1, 0
G ^F N ^E		p, p	-p, 0
N ^F G ^E		1 - p, p - 1	p - 1, 0
N ^F N ^E		0, 0	0, 0

Figure: The gift game in Bayesian normal form

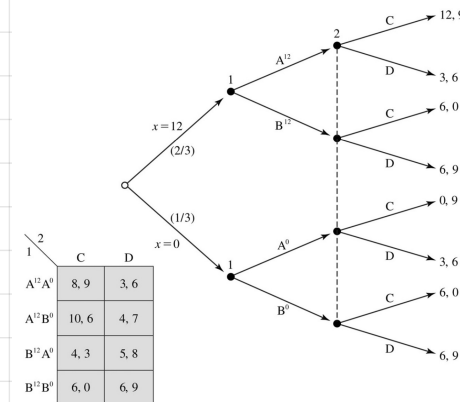


Figure: Extensive-form and normal-form representations

First Method: Bayesian NE and Rationalizability

- $A^{12} A^0$ and $B^{12} A^0$ are dominated
- C is dominated.
- $A^{12} B^0$ is dominated
- The Bayesian rationalizable set is thus $\{(B^{12} B^0, D)\}$.

Second Method

Consider a simple Cournot Duopoly game with incomplete information.

- Suppose that demand is given by $p = 10 - Q$, where Q is the total quantity produced in the industry.
- Firm 1 selects a quantity q_1 , which it produces at zero cost.
- Firm 2's cost of production is private information (selected by nature).
 - With probability 1/2, firm 2 produces at zero cost, L type.
 - With probability 1/2, firm 2 produces at a marginal cost of 4, H type.
- Firm 2 knows its type, whereas firm 1 knows only the probability that L and H occur.

Let q_2^H and q_2^L denote the quantities selected by the two types of firm 2. The payoff functions are the following:

$$\begin{aligned}
 \bullet U_1^L &= (10 - q_1 - q_2^L) q_1^L \\
 \bullet U_1^H &= (10 - q_1 - q_2^H) q_1^H - 4 q_1^H \\
 \bullet U_1 &= \frac{1}{2} (10 - q_1 - q_2^L) q_1 + \frac{1}{2} (10 - q_1 - q_2^H) q_1
 \end{aligned}$$

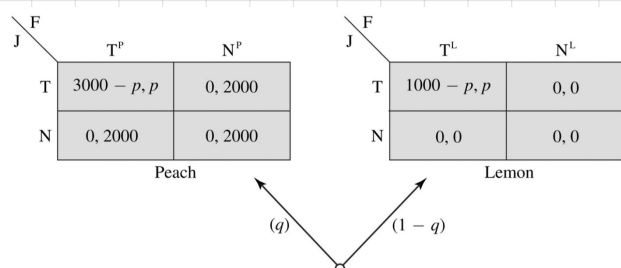
Find the best-response functions for the **three** players.

$$\begin{aligned}
 \bullet BR_1(q_2^L, q_2^H) &= 5 - \frac{1}{4} (q_2^L + q_2^H) \\
 \bullet BR_2^L(q_1) &= 5 - \frac{1}{2} q_1 \\
 \bullet BR_2^H(q_1) &= 3 - \frac{1}{2} q_1
 \end{aligned}
 \longrightarrow q_1 = 4, \quad q_2^L = 3, \quad q_2^H = 1$$

Lemon

Jacob is in the market for a used car. He meets a shifty looking man named Francesco at Mercatino delle Occasion who has an attractive 15-years-old car for sale. He imagines himself at the wheel, cruising in Milan, taking in the flirtatious glances of many woman's through his sunglasses. Then he imagines the engine exploding, followed by an embarrassing scene in which he watches from the curb as a firefighter dowsed his vehicle with water. Jacob says to Francesco, "The car looks good, but how do I know it isn't a lemon?" Francesco rejoins, "Bro, you have my word, this care is a peach; it's in great shape!" Jacob insists, "nonsense! Since I will not likely see you again, your word means nothing to me. I know game theory. You have no way of establishing a reputation with some one you will never meet again." Francesco nods and says, "Okay, lets talk turkey. KBB tells the market price. In the end, you will have to decide whether you are willing to pay the Blue Book price for the car, as I must decide whether to offer the car at this price."

Suppose there is some fixed market price, p , for fifteen-years-old sedans of the type that Francesco is selling. Assume that Jacob and Francesco play the game as follows:



What are the equilibria?

Two kinds of equilibria are possible in this game. In the first kind, only the lemon is traded. The second kind of equilibrium features trade of both the lemon and the peach.

Only Lemon Traded

Let us check whether there are values of p for which only the lemon is traded in equilibrium:

- Jacob selects T and Francesco plays the strategy $N^P T^L$ (no trade if peach; trade if lemon).
- In order for Francesco's strategy to be optimal, it must be that $0 \leq p \leq 2000$.
- Conditional on knowing that Francesco only offers the lemon for sale, Jacob is willing to trade: $q \cdot 0 + (1-q) \cdot (1000 - p) \geq 0 \Leftrightarrow p \leq 1000$

Both Traded

Let us check conditions for which only the lemon is traded in equilibrium:

- Jacob selects T and Francesco plays the strategy $T^P T^L$ (no trade if peach; trade if lemon).
- p must be high enough so that Francesco is willing to sell the peach, $p \geq 2000$.
- Jacob's expected value of owning the car must be at least as great as the price: $q \cdot 3000 + (1-q) \cdot 1000 \geq p \Leftrightarrow 1000 + 2000q \geq p$.
- There is an equilibrium in which both types of car are traded as long as: $1000 + 2000q \geq p \geq 2000$.
- $q \geq 1/2$ is a necessary condition.
 - Unless the probability is sufficiently high (there are not too many lemons in the world), there is no equilibrium in which the peach is traded.
 - The price $p \in [2000, 1000 + 2000q]$
- If $q < 1/2$, then only lemons are traded in equilibrium.
 - This outcome is inefficient because trading the peach creates value.
 - Thus, the model demonstrates that asymmetric information sometimes causes markets to malfunction.

Auction

In the lemons example, the seller has private information. Other markets have different information structures. In many instances, prospective buyers have private information. Seller would like to find the buyer with the highest willingness to pay and strike a deal with this buyer at a price close to the buyer's valuation. (Un)fortunately, for the seller, she may not know the prospective buyers' valuations. One way for the seller to encourage competition between prospective buyers and to identify the highest valuation is to hold an auction.

- Nowadays, objects worth astronomical sums of money change hands every day in auctions, not only household objects, art and antiques, but also securities, minerals and energy.
- Public procurements can also be conducted as auctions.

A Simple Auction Environment

The seller has a painting worth nothing to her. Two potential buyers, 1 and 2. Buyer i 's valuation $v_i \sim U[0, 900]$ and is independent with v_j . If buyer i gets the painting and pays x , his utility is $v_i - x$; If buyer i doesn't get the painting and pays x , his utility is $-x$. Each buyer i privately knows his own v_i but not v_j . $U[0, 900]$ is a common prior among buyers and the seller.

Second-Price Sealed-Bid Auction

Each buyer / bidder simultaneously and independently submit bid b_i . The painting is awarded to the highest bidder. In case of tie, randomly award with equal probability. The winner pays a price equal to the second-highest bid.

- For example, $b_1 = 900$, $b_2 = 200$.
- Buyer 1 wins, and pays 200.
- Buyer 1's payoff is $v_1 - 200$; Buyer 2's payoff is 0.

Suppose you are buyer i with value 500. You do not know how to bid. You ask your dear friends for advices.

- Your friend A proposes bidding 500 because being honest is a merit.
- Your friend B proposes bidding 400 so that you pay less when you win.

Let consider four cases:

- 1) $b_j > 500$. $u_i(500, b_j) = 0 = u_i(400, b_j)$
- 2) $b_j = 500$. $u_i(500, b_j) = 0 = u_i(400, b_j)$
- 3) $400 \leq b_j < 500$. $u_i(500, b_j) = 500 - b_j > u_i(400, b_j)$
- 4) $b_j < 400$. $u_i(500, b_j) = 500 - b_j = u_i(400, b_j)$

Your friend T who have taken game theory classes tells you it means 500 weakly dominates 400 in the language of GT.

Your friend B has a new proposal of bidding 600 so that it is more likely to win. Shall you follow his new advice? Let's compare 600 and 500 by considering four cases:

- 1) $b_j \geq 600$. $u_i(500, b_j) = 0 = u_i(600, b_j)$
- 2) $500 < b_j < 600$. $u_i(500, b_j) = 0 > b_j - 500 = u_i(600, b_j)$
- 3) $b_j = 500$. $u_i(500, b_j) = 0 = u_i(600, b_j)$
- 4) $b_j < 500$. $u_i(500, b_j) = 500 - b_j = u_i(600, b_j)$

Your friend T says you can generalise this observation to any bid higher than 500.

500 weakly dominates any $x > 500$. Let compare 500 and x by considering four cases:

- 1) $b_j \geq x$. $u_i(500, b_j) = 0 = u_i(x, b_j)$
- 2) $500 < b_j < x$. $u_i(500, b_j) = 0 > b_j - 500 = u_i(x, b_j)$
- 3) $b_j = 500$. $u_i(500, b_j) = 0 = u_i(x, b_j)$
- 4) $b_j < 500$. $u_i(500, b_j) = 500 - b_j = u_i(x, b_j)$

Similarly, you can show 500 weakly dominates any $x < 500$, friend T argues.

Buyer i 's strategy: $b_i: [0, 900] \rightarrow \mathbb{R}$. $b_i(v_i)$: buyer i 's bid when his value is v_i . Claim: Bidding true valuation, $b_i(v_i) = v_i$, is a weakly dominant strategy for each player.

A weakly dominant strategy is one that weakly dominates all other strategies, so it is a best response to every strategy of the other player.

S_i weakly dominates S_i' if: 1) $u_i(S_i, S_{-i}) \geq u_i(S_i', S_{-i})$ for all S_{-i} 2) $u_i(S_i, S_{-i}) > u_i(S_i', S_{-i})$ for some S_{-i} .

First-Price Sealed-Bid Auction

Each buyer/bidder simultaneously and independently submit bid b_i . The painting is awarded to the highest bidder. In case of tie, randomly award with equal probability. The winner pays a price equal to the first-highest bid.

- For example, $b_1 = 900$, $b_2 = 200$
- Buyer 1 wins, and pays 900.
- Buyer 1's payoff is $v_1 - 900$; Buyer 2's payoff is 0.

You turn to your friend T for help again. He says the following:

- Do NOT bid more than your valuation. You would then get a negative payoff if you win.
- Bid your valuation gives you zero payoff.
- You should consider bid less than your valuation, $b_i(v_i) < v_i$. You would then get a positive payoff if you win.

It turns out the other buyer also went to T for help. He recommends both of you bidding half of your valuation, $b_i(v_i) = v_i/2$. Shall you follow his advice?

Suppose your value is v_i and you are choosing an optimal bid x . If you win, your payoff will be $v_i - x$. You win if $x \geq b_j = v_j/2$, then $P(v_j/2 \leq x) = P(v_j \leq 2x) = 2x/900$.

Your expected payoff of bidding x is: $(v_i - x) \times x/450 \Rightarrow x^* = v_i/2$.

Indeed, $b_1(v_1) = v_1/2$ and $b_2(v_2) = v_2/2$ constitute a BNE.

Recall that the winner of the second-price auction pays the second-highest bid, which (as you can check) happens to be one-half of the winner's valuation on average. Thus, the first-price and second-price auctions yield the same expected revenue for the seller, 300. This about "revenue equivalence" between different auction forms actually holds in many auction environments.

Perfect Bayesian Equilibrium

- Rationalizability, NE, and SPE in games with complete information.
- Bayesian Rationalizability, Bayesian NE with incomplete information.
- Bayesian SPE? Not really...
- The next example shows we need something goes beyond subgame perfection.

A Variant of the Gift Game

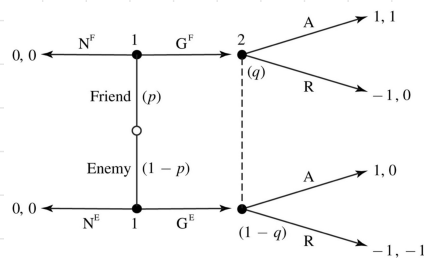


Figure: A Variant of The Gift Game

- P2 enjoys opening gifts from both types of P1.
- The game has sequential decisions.
- Subgame perfection? No proper subgame.
- Every BNE is subgame perfect, in particular $(N^F N^E, R)$ is.
- However, R is dominated by A.
- We need a solution concept that **isolate every information set** for examination (where as SPE isolates every subgame).

Suppose $p = 1/2$ and you are player 2.

- Suppose you know P1 is playing $N^F G^E$.
- A gift is presented to you. Is it from a friend or enemy? Enemy

Suppose $p = 1/2$ and you are player 2.

- Suppose you know P1 is playing $G^F N^E$.
- A gift is presented to you. Is it from a friend or enemy? Friend.

Suppose $p = 1/2$ and you are player 2.

- Suppose you know P1 is playing $(q_1 = 0.5, q_2 = 0.5)$ where q_1 and q_2 are the probability of N^F and N^E .
- A gift is presented to you. What is your belief about the gift?

The gift game: Belief

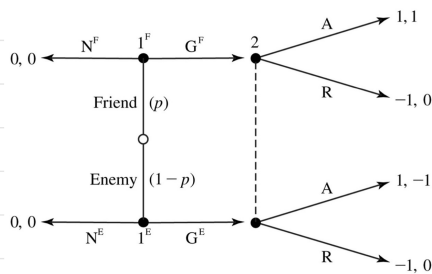


Figure: The gift game

Sequential Rationality

Suppose $p = 1/2$ and you are player 2.

- Suppose you know P1 is playing $(q_1 = 0, q_2 = 0.5)$ where q_1 and q_2 are the probability of N^F and N^E .
- A gift is presented to you. What is your belief about the gift?

Suppose $p = 1/2$ and you are player 2.

- Suppose you know P1 is playing $N^F G^E$.
- A gift is presented to you. It is from Enemy. Will you accept or reject? R.

Suppose $p = 1/2$ and you are player 2.

- Suppose you know P1 is playing $G^F N^E$.
- A gift is presented to you. It is from Enemy. Will you accept or reject? A.

Suppose $p = 1/2$ and you are player 2.

- Suppose you know P1 is playing ($q_1 = 0.5, q_2 = 0.5$) where q_1 and q_2 are the probabilities of N^F and N^E .
- A gift is presented to you. $q = 1/2$
- Will you accept or reject? Indifferent.

Suppose $p = 1/2$ and you are player 2.

- Suppose you know P1 is playing ($q_1 = 0, q_2 = 0.5$) where q_1 and q_2 are the probabilities of N^F and N^E .
- A gift is presented to you. $q = 2/3$
- Will you accept or reject? A.

PBE

Definition: Amusement

An amusement is pair of (α, μ) , where α is a strategy profile and μ is a belief system; i.e., for each information set I , $\mu(\cdot|I)$ is a probability measure over nodes in I .

Definition: Perfect Bayesian Equilibrium (PBE)

An amusement (α, μ) is a weak Perfect Bayesian Equilibrium (WPBE) if at each information set I the following are satisfied:

- 1) Sequential Rationality. given belief system μ , α is sequentially rational.
- 2) Bayesian Updating. given strategy profile α , μ is consistent with Bayesian rule whenever possible.

Finding PBE

1. Start with a strategy for P1.
2. Calculate posterior (updated beliefs) by using Bayes' rule.
 - In the event that Bayes' rule cannot be used, arbitrarily select a posterior and you will generally have to check different potential posteriors with the next steps.
3. Given the posterior, calculate P2's optimal action.
4. Check whether P1's strategy is a best response to P2's strategy.
 - If so, you have found a PBE.

Two special classes of (pure-strategy) equilibria.

- Separating equilibrium: the types of a player behave differently.
- Pooling equilibrium: the types behave the same.

PBE: The gift game

There are 4 potential pure-strategy equilibria:

- 1) Separating equilibrium: $(N^F G^E)$ with a consistent belief q .
- 2) Separating equilibrium: $(G^F N^E)$ with a consistent belief q .
- 3) Pooling equilibrium: $(G^F G^E)$ with a consistent belief q .
- 4) Pooling equilibrium: $(N^F N^E)$ with a consistent belief q .

PBE: The gift game - $(N^F G^E)$

- 1) P1 chooses $(N^F G^E)$
 - 2) By Bayes' rule, $q = 0$.
 - 3) given $q = 0$, P2's optimal strategy is R.
 - 4) given R, P1's optimal strategy is $(N^F N^E)$
- NO PBE

PBE: The gift game - $(G^F G^E)$

- 1) P1 chooses $(G^F G^E)$
 - 2) By Bayes' rule, $q = p$
 - 3) given $q = p$, P2's optimal strategy is:
 - If $p < 1/2$, P2 R; given R, $(N^F N^E)$
 - If $p = 1/2$, P2 A; given A, $(G^F G^E)$
 - If $p = 1/2$, P2 is indifferent between R and A. Only A supports a PBE.
- A PBE is $((G^F G^E), A, q = p)$ when $p \geq 1/2$.

PBE: The gift game - $(G^F N^E)$

- 1) P1 chooses $(G^F N^E)$
 - 2) By Bayes' rule, $q = 1$.
 - 3) given $q = 1$, P2's optimal strategy is A.
 - 4) given A, P1's optimal strategy is $(G^F G^E)$
- NO PBE

PBE: The gift game - $(N^F N^E)$

- 1) P1 chooses $(N^F N^E)$
 - 2) Bayes' rule has no bite, $q \in [0, 1]$.
 - 3) In order to support $(N^F N^E)$, P2's strategy should be R.
 - 4) In order to support R, P2's belief $q \leq 1/2$.
- A PBE is $((N^F N^E), R, q \leq 1/2)$ for any p .