Principles of Finance – Stefano Rossi

Introduction

What is Finance? & What is the Point of a Finance Course?

- Method and Language: How do corporations, investment banks, consulting firms, and academic financial economists think?
- What language are these people speaking?
    A. For example, what do they mean by NPV, WACC, and a thousand other impenetrable phrases?
- What is all this stuff in the WSJ or The Economist or the Financial Times or the Sole24Ore about?

How do Executives, Investors & Bankers Decide?

Finance is about **valuation**.

- How can you convince your boss to fund a project?
- How can you convince an employer to hire you, i.e., that you are worth the money?
- How can you/corporations get money from investors and elsewhere? What is the price of such investments? What is a financial claim and how should you think of them?
- Should you/corporations get money? For what kind of projects?
- Where should you invest your money? What is a good reward for risk?
- How do you add value? What exactly is value?

Until you learn how to estimate value, you cannot answer the preceding questions.

Finance and Big Data

- **Finance uses Big Data**, prices of shares and other financial assets are among the most easily accessible Big Data in the world.
- **Finance Is Big Data**
    A. Finance as an academic discipline was born in conjunction with availability of large data on asset prices: Center for Research on Security Prices (CRSP) established in 1960 at University of Chicago.
    B. First academic study on rates of return of investments in common stock by Lawrence Fisher and James H. Lorie, based on 2,000,000 data entries and 400,000 stock price observations over 1926 – 1960 , back then such data was "big".
    C. Eugene F. Fama's PhD thesis established statistical properties of stock prices and returns.
    D. Theories such as Capital Asset Pricing Model (CAPM) based on these data; today, trillions of observations are available.
- Where do you start?
    A. First, you need to know how to think about finance, and which questions to ask the data., and this is what this course is about.
    B. After that, if you like what you see, you might want to dig deeper and analyze data on asset prices and returns.

Valuation

Basic Purpose of Finance - Valuation

- **Capital Budgeting**
    A. Should you undertake a particular project?
    B. What is this project worth?
- **Investments**

    A. How should investors allocate their money across many projects in their investment portfolios?
- **Capital Structure**
    A. To undertake a project, should you borrow money, sell shares to partners, or finance it yourself?
    B. Should firms finance specific projects or disperse funds?

Background Disciplines
To answer these questions, Finance is a hybrid of:
- **Economics** (the science of choosing among tradeoffs)
- **Statistics** (the science of dealing with uncertainty)
- **Accounting** (the language of business).

What is Valuation?
Valuation is forward looking, not backward looking.
- Value today depends on net cash flow from now to eternity.
- History can matter: path of future cash flows can depend on today's "stock" of your assets.
- Historical experience can help you judge the future.

Finance cannot estimate philosophical or moral value.
- However, finance can deal with such values **if you** are willing to translate them into monetary values.
    A. E.g., if you value your being honest at $1 million, you can then use financial tools to analyze your decisions.
    B. Attaching such values is obviously subjective, not objective.

Law of One Price – Relative and Absolute Value
- All financial valuation is relative to alternatives.
- The "**Law of One Price**" – Same thing should cost the same.
- A natural extension – Similar things should cost similarly.
- Relative valuation can be easy or hard, depending on closeness to alternatives.

Easy Valuations
- One share of IBM vs. another share of IBM.
- One option of IBM vs. one share of IBM.
- One share of PepsiCo vs. one share of Coca Cola.
- One share of PepsiCo vs. one share of IBM.

Hard Valuations
- The White House vs. The Coliseum.
- A Nuclear Power Plant vs. Vacation Time.
- Mars Exploration vs. Treasury Bonds.

A Project
A project is anything that generates a series of cash flows.
- A repair shop. Cost: $500; Expected earnings: $1,000 per month, starting next month, for 24 months.
- A company's environmental spill cleanup project, Cash flows: −$20, 000 per year for 5 years.
- An investor purchases a Certificate of Deposit (CD) from a bank for $10,000; The bank will repay $12,000 in two years.
- An education. It costs $50,000, foregoes salary for 3 years, pays off (an additional) $20,000 in salary for 30 more years.

- A lottery ticket that costs $1, and pays $14,000,000 with some prob.

Projects range from physical investments (bicycle repair shop), to pure monetary investments (CD), to pure gambles (lottery ticket). Finance cares mostly about cash flows, whatever their source.

- How cash flows are generated concerns other disciplines (e.g., production or marketing).

Bonds (Debt) and Stocks (Equity)

Two particularly common projects: Stocks & Bonds.

- From the perspective of finance and valuation, these two claims are (primarily) sets of cash flow streams paid to holders of these claims.
- Bonds (usually) promise fixed payments at fixed points in time.
  - A. Thus, they are often called "Fixed Income".
  - B. They are also often safer, but they don't always pay out what they promise.
- Stocks (usually) get what is left over after the bonds have been paid off. Stock is sometimes called "equity" or "levered equity".
- Debt and equity are often called financial securities.
  - A. Strictly speaking, security means "registered under the 1933 U.S. Securities Act", but we will also use this name for unregistered, privately-held, and non-U.S. claims.

Jargon – Bonds (Debt)

A **Bond** is a contractual obligation by a borrower to pay certain amounts of cash in the future to a lender.

- A 30-year fixed mortgage loan is a good example of a bond: the homeowner borrower promises the bank lender to pay a fixed amount of cash every month. The homeowner issues a bond to the bank.

A bond is a particular type of Loan:

- Bonds were traditionally and many bonds still are fixed-rate loans ("fixed income" instruments): they have future payments which are fixed and thus do not change with the interest rate.
- How often can your bank savings deposit's interest rate change?
- What kind of bond (from bank to you) is a checking account?
- We often think of bonds as something with fixed promised payments in the future – a special kind of loan.

A Firm

A firm is a collection of projects, financed by claims, that provide the inflows that eventually should generate outflows.

- A firm or company invests in various projects.
- These projects generate expenses and produce revenues.
  - A. Firms are much simpler in finance than they are in other disciplines! In a sense, they are a "black box" that produces cash flows.
  - B. Do not worry, the situation will quickly become more than complex enough.
- Revenues in excess of expenses go either into new investments or back to the firm's claimants.
  - A. Claimants can be debt and equity holders.
  - B. Other claimants can be suppliers who sold product on credit, or government who demands income tax, sales tax, etc.

Corporate Accounting Identities – I

By definition, firm value is the sum of the value of all claims.

- Firm = Debt (Loans) + Other Liabilities + (Levered) Equity (Owners) = All Future Payouts.

If you own all claims, you own the firm!

If the value of the firm's assets (and future opportunities) were not equal to the value of all its securities, you could get rich easily.

- At least if the market is close to perfect.

If you purchase all claims of the projects today, you own the project.

- Firm value = Value of all claims

Firm value is also all current and future net flows.

- If you own the project, you own all net earnings of the project
- Firm value = value of all current and future project-generated net cash flows.

If you own the project, you receive all net payouts of the project.

- Firm value = value of all current and future payouts (dividends, interest, tax payments, etc.)

Moving Flows Across Time

The time distribution of future inflows or outflows can be shifted neutrally in a perfect market, properly accounted for:

- For example, you can pay out D in dividends today, or D (1+r) in dividends tomorrow – the project value remains the same.
- Whether cash flows (or earnings) grow or shrink, or whether dividends are zero today or zero next year or growing or shrinking is irrelevant. All that matters is the project's total present value (PV) of al future net cash flows today.
- You can reduce cash flows today to jack them up tomorrow (i.e., by reinvesting them at rate r). You can increase dividends today by $1 at the expense of $(1+r) dividends tomorrow. It makes no difference to project's PV.

Takeaways

Finance is about valuation of cash flows

- Value of project = Present value of all its future net cash flows.
- Value of claim = Present value of all of its future net cash flows.
- Value of firm = Value of all claims on it (often just debt + equity)

Law of One Price – Same things should cost the same.

Time Value of Money

Perfect Markets

For now, and the next few sessions, we pretend we live in a perfect market. A perfect market must satisfy four assumptions:

- **No differences in opinion**, uncertainty is ok, but everyone must agree to exactly what it is. We must not have different information or opinions.
- **No taxes**, or any government interference or regulation (except perfect property rights).
- **No transaction costs**, neither direct nor indirect.
- **No big sellers/buyers**
    A. There must always be more where they came from.
    B. No (few) investors or firms are special.
    C. If investors differ, there must be infinitely many clones competing.

Why? What do Perfect Markets do for you?

It is the simplest world. Analysis is easiest. Life is tough enough even in a perfect market.

Any logic that fails in this simplest of worlds will be wrong in a more realistic world, too.

- Put differently, as your world gets closer and closer to perfection, your methods need to become closer and closer to what must hold in a perfect market.

Further Assumption, Only for Starting Exposition

Today, we further assume perfect certainty.

- We know what the rates of return on every project are and will be.

- We don't need to worry about statistics and attitudes towards risk.
- All same-period (interest) rates of returns are the same.

Today, we also assume equal rates of returns per period.
- No worries about yield curve.
- A 1-year bond offers the same annualized rate of return as a 30-year bond.

Notation and Definitions
Time Convention:
- 0 = Today
- 1 = Next period
- t = some time period (in the future)
- T = often to denote a final time period

Flow is something accumulating over a time span, e.g., a rate of return from t-1 to t. Opposite is **Stock**, a quantity at some moment
- C or CF = Cash Amount.
- C_t = instant cash amount at time.
- $D_{t-1,t}$ = a flow of D (e.g. dividends) over a time period t-1 to t.
- D_t = common casual notation for $D_{t-1,t.}$
- $D_{15,20}$, a flow of D (e.g. dividends) from time 15 through 20.
- Return vs Net Return vs. Rate of Return (Interest Rate).
- r, r_1, $r_{15,20}$, r_8: Rates of Return.

Small Notes
This is a bit inconsistent. Dividends are really also paid at one instant in time, and thus should not be subscripted like a flow. If the investment is a loan, the rate of return is usually called an interest rate. We will (almost always) use the name "rate of return" and "interest rate" interchangeably.
Although there is a difference between a return (CF_1), a net return ($CF_1 - CF_0$), and a (net) rate of return ($(CF_1 - CF_0)/CF_0$), in conversations, it is rarely explicit. Usually, you are assumed to know what the speaker means.

Rate of Return
Rate of return from investing CF_0 today and getting CF_1 at t = 1 is:

$$r = r_{0,1} = \frac{(CF_1 - CF_0)}{CF_0} = \frac{CF_1}{CF_0} - 1$$

This is the main formula of finance. With dividends D (or coupons or rent) paid at the end of the period (thus not reinvestable):

$$r = r_{0,1} = \frac{(CF_1 + D_{0,1} - CF_0)}{CF_0} = \frac{(CF_1 + D_{0,1})}{CF_0} - 1$$

Using our abbreviations,

$$r = r_1 = \frac{(CF_1 + D_1 - CF_0)}{CF_0} = \frac{(CF_1 + D_1)}{CF_0} - 1$$

- **Dividend** (or coupon or interim payment) **Yield** is $\frac{\boldsymbol{D_{0,1}}}{\boldsymbol{CF_0}}$

- **Capital Gain** is $CF_1 - CF_0$
- **Percent Price Change** is $\frac{(CF_1 - CF_0)}{CF_0}$
- **Total Rate of Return** is **Percent Price Change** plus **Dividend Yield**.

If I casually write $r = \frac{P_1}{P_0} - 1$ for the rate of return, then I am assuming that P_1 includes any interim payments.

Discussion Questions

- If the rate of return is positive, can the percent price change be negative?
  Yes, as long as dividend yield is positive and greater than the percent price change.

- If you invest $5 and will receive $8 in 10 years, what is your (holding) rate of return?
  (8-5)/5 = 60%

- Can a rate of return be negative? Can an interest rate be negative?
  Yes & No

- If you have $5 and earn a rate of return of 250%, how much money will you have?
  $5 * (1 + 2.5) = $17.5

- If you have $5 and earn a rate of return of 40%, how much money will you have?
  $5 * (1 + 0.4) = $7

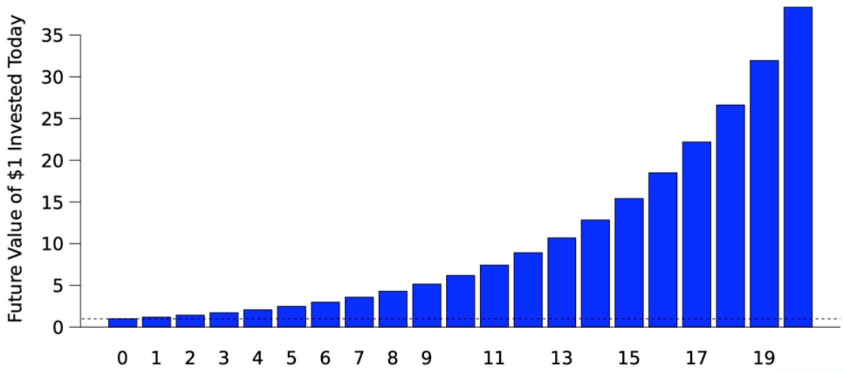
- Compare 10% to 5%. Which of the following is true?
  A. Would you say that 10% is 5% more than 5%?
  B. Would you say that 10% is 100% more than 5%?

- If you invest $55,000 at an interest rate of 350 basis points above the 5% interest rate, what will you receive at the end of the period? $55,000 * (1 + 0.05 + 0.035) = $59,675

- If you have $5 and you earn a rate of return of 20% in the first year and a rate of return of 20% the following year, how much money will you have? $5 * (1 + 0.2)^2 = $7.20

- If you have $5 and you earn a rate of return of 20% in each year, how much money will you have in x years? $5 * (1 + 0.2)^x

Compounding at 20% Rate of Return Per Year



- You have $100. You invest half in two firms. Firm 1 makes 10% this month. Firm 2 makes 20% this month. How much did your portfolio make in total? 115$

- If the 1-month interest rate is 1%, what is the 1 year rate? (1 + 0.01)ˆ12 = 1 + 12.68%

- If the 1-day rate is 0.02%, what is the 7-day (weekly) rate? (1 + 0.0002)ˆ7 = 1 + 0.14%

- A project for $200 promises to return 8% per year. How much will you have after 1 month? R_m = (1 + 8%)ˆ(1/12) – 1 = 0.64%. So, $200 + (1 + 0.64%) = $201.3

- If the annual interest rate is 14%, what is the daily interest rate? (1 + r_{daily})ˆ365 = (1 + r_{yearly}) r_{daily} = (1 + r_{yearly})ˆ(1/365) = (1 + 0.14)ˆ(1/365)

- The annual interest rate is 14%, what is the weekly interest rate? 365/7 = 52.14285714. So, (1 + 14%) = 0.2516031

Some Useful Algebra

$$x^a = b \iff x = b^{1/a}$$

$$a^x = b \iff x = \frac{\log b}{\log a}$$

Rule of Thumb
If both the interest rate and the number of time periods is small,

$$(1 + r_n)^t \approx 1 + t \times r_n$$

Adding up instead of compounding gets to be a worse approximation if time increases and if the interest rate increases. It also matters how much money is at stake.

Warning: Conventions and Jargon
In principle, interest rates (and quotes) are not difficult, but they are tedious and often confusing, because everyone computes and quotes them slightly differently.
Sometimes, it is obvious what people mean, sometimes interest rates are intentionally obscure in order to deceive you.
- You should know what you are talking about.
- Ask if you are unclear! There can be a lot of money at stake.
- Arbitrage desks on WS make most of their money on spreads below 20 basis points.

- A bank quotes you 8% interest per year. If you invest $1 million in the bank, how much will you end up with? 8%/365 = (1 + 8%/365)ˆ365 – 1 = 8.33% (Annual Interest Rate)

- So is the 8% posted by the bank a true annual interest rate? No

Interest Quotes (Not Rates)
Unfortunately, many institutions give you interest "quotes", rather than interest rates, and the two are easy to confuse.
- This is especially bad with annualized interest quotes.
- There are many "pseudo interest rates" which are really "interest quotes" and not true "interest rates".

For example, sometimes banks and lenders calculate and pay daily interest rates, although they only credit the interest payment to accounts once per month. Such banks' daily interest rate calculation is the quoted annual interest rate divided by 365 ($r_d = r_{y/365}$). Note that some banks use 360 days. Some banks quote for clarity: Interest rate: 8% compounded daily. Effective annual yield: 8.33%.

Quotes vs. Rates: Government Bonds

At US Treasury Auctions, the government sells Treasury bills that pay$10,000 in 180 days.

If the government discount quote is 10% (which is absolutely not an interest rate), then it means you can purchase the Treasury bill at the auction for $9,500, because they use the formula:

$$TB\ Price = \$100 * \left[100 - \left(\frac{(days\ to\ maturity)}{360}\right) * discount\ quote\right]$$
$$= \$100 * \left[100 - \left(\frac{180}{360}\right) * 10\right] = \$9{,}500$$

Financial newspapers and websites often print "95" instead of "9,500" because it is shorter, and T-bills are quoted in units of 100.

- How does the price of a bond change if the economy-wide interest rate changes? It changes

- If you will receive $7 next year and another $7 in two years, and the prevailing (alternative) interest rate [or cost of capital] is 40%, do you have the equivalent of $14?

- Suppose you will receive $7 next year and another $7 in two years, and the prevailing (alternative) interest rate [or cost of capital] is 40% per year.
    A. What would you value this project as of today?
       PV = $7/(1 + 0.4) + $7/(1 + 0.4)^2 = $8.57
    B. What formula are you using? PV & NPV
    C. If this project costs $8, should you take this project? Yes, because it's NPV = PV – Cost > 0.

Net Present Value

Given Cash Flow CF_t at t, What is the PV Formula?

The present value of cash CF_t at time t is:

$$PV = \frac{CF_t}{(1 + r_{0,t})}$$

The quantity $\frac{1}{(1+r_{0,t})}$ is called the **discount factor**. It is the facto that is multiplied to a future cash flow in order to obtain the future cash flow's current value.

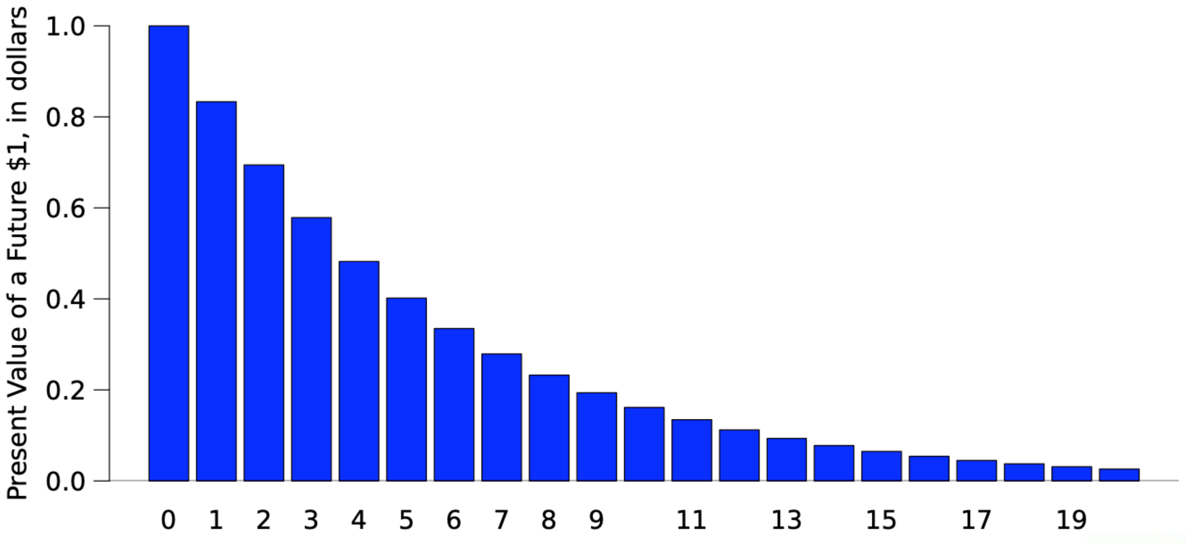
The quantity $r_{0,t}$ is called the **discount rate**, because it is the interest rate that is used to obtain the discount factor.

In this context, the discount rate is also often called the **(opportunity) cost of capital**, because you should think of it either representing your alternative investment opportunities (if you have money) or your cost of borrowing (if you need money).

In our perfect market, the two are the same. That is, in our financial markets, you can invest into infinitely many alternatives for a rate of return that is exactly equal to your cost of borrowing.

NPV simply means including the time-0 cash flow, often a cost (negative).

Discount at 20% Rate of Return Per Year



NPV Formula
The net present value is:

$$NPV = CF_0 + \frac{CF_1}{(1 + r_{0,1})} + \frac{CF_2}{(1 + r_{0,2})} + \cdots = \sum_{t=0}^{\infty} \frac{CF_t}{(1 + r_{0,t})}$$

It is called "net", because the first cash flow CF_0 is often negative.
In a perfect world, you should take all positive NPV projects. This is called **the NPV capital budgeting** rule.

The Logical Foundation of the NPV Rule
Here is how a perfect world without uncertainty must work:
- The NPV rule is optimal (other rules leave money on the table), and
- Positive NPV projects must be scarce.

The argument in more detail and its proof:
Assume that, in our perfect market, you can borrow or lend money at 8% anywhere today. NPV formula says you won't make money on projects that cost $1 today and yield $1.08 next year. It says you should take all projects that yield > $1.08 next year.
Now, assume you have (infinite) many investment opportunities that cost $0.99 and yield $1.08. (NPV is positive). How would you get rich? Borrow $0.99 and use it to buy the project. Tomorrow, you pay $1.07, and receive $1.08. You earn $0.01. If you prefer money today, borrow against the $0.01, or borrow $1.00.
- If such projects are in limited supply, you (and everyone else) would buy up all such projects, until the project's equilibrium price has increased to make the project zero NPV.
- (If you can short projects, and you have willing buyers for negative-NPV projects, you can sell them: invert the argument).

Takeaways
Total rate of return is percent price change plus dividend yield.
Net Present Value: Definition & Capital Budgeting Rule.

Perpetuities and Annuities

Maintained Assumptions

Today, we maintain the assumption of perfect markets, so we assume four market features:
- No differences in opinion
- No taxes
- No transaction costs
- No big sellers/buyers – infinitely many clones that can buy or sell

We again assume perfect certainty, so we know what the rates of return on every project are.
We again assume equal rates of returns in each period (year).

General Questions

- Are there any shortcut NPV formulas for long-term projects – at least under certain common assumptions?

- Or, do we always have to compute long summations for projects with many, many periods?

- Why do some of the folks in the room have the ability to quickly tell you numbers that would take you hours to figure out?

- How are loan payments (e.g., for mortgages) computed?

- If your firm produces \$5 million/year forever, and the interest rate is a constant 5% forever, what is the value of your firm?

- If your firm produces \$5 million/year in real (inflation-adjusted) terms forever, and the interest rate is a constant 5% forever, what is the value of your firm?

- What is the value of a firm that generates \$1 in earnings per year and grows by the inflation rate?

- What is the monthly payment on a 6% 30-year fixed rate mortgage?

- Perpetuities and Annuities are shortcut formulas that can make computations a lot faster, and whose relative simplicity can sometimes aid intuition.

Perpetuities

Simple Perpetuities

A perpetuity is a financial instrument that pays C dollars per period, forever. If the interest rate is constant and the first payment from the perpetuity arrives in period 1, then the PV of the perpetuity is:

$$PV = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

Make sure you know when the first cash flow begins: Tomorrow [t = 1], not today [t = 0]!

Proof of PV Perpetuity

The formula is:

$$\frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^t} = \frac{C}{r}$$

We want to prove that the above formula is correct. Divide all terms by C and obtain:

$$\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^t} = \frac{1}{r}$$

Multiply the above by (1 + r) and obtain:

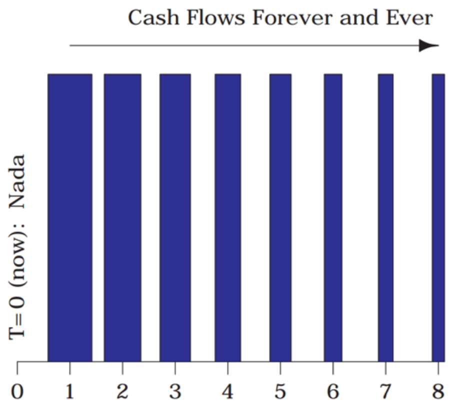
$$1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{C}{(1+r)^{t-1}} = \frac{1+r}{r}$$

Subtract the second line from the third line:

$$1 = \frac{1+r}{r} - \frac{1}{r}$$

Whereby the RHS simplifies to r/r, which equals 1 and prove the statement.

How can an Infinite Sum Be Worth Less Than ∞?
Because each future cash flow is worth a lot less than the preceding cash flow. In the graph, the PV of each cash flow is the bar's area.



Note: Time Consistency
Is the formula time-consistent? For example, if my house/property is paying up \$100 eternally, and I get cash, how can it still be worth the same tomorrow as it is today?
- The question is: if you have a perpetuity worth \$1,000, you will still have an annuity worth \$1,000 next year and get one payment, too. How can this be?

Next year (year 1), you will still have a perpetuity (beginning year 2). How much will the perpetuity be worth next year?
- Presume a cash flow of \$10 each year.
- Presume the interest rate is 10%.
- The perpetuity is thus worth \$100
- Now, consider standing tomorrow
    A. You will still own a perpetuity
    B. It will then be worth \$1,000 – but this is tomorrow.
    C. Today's value of tomorrow's perpetuity: \$1,000/(1 + 10%) = \$909
- In addition, you get one extra cash flow of \$100 tomorrow, which is worth \$91.

Growing Perpetuities
A growing perpetuity pays CF, then CF(1+g), then CF(1+g)ˆ2, …

For example, if CF = \$100 and g = 10%, then you will receive the following payments:

$$
\begin{array}{lll}
CF_0 = 0 & & \\
CF_1 = 100 & & \text{then discount with } r_{0,1} \\
CF_2 = 100\cdot(1+10) & = 110.00 & \text{then discount with } r_{0,2} \\
CF_3 = 100\cdot(1+10)^2 & = 121.00 & \text{then discount with } r_{0,3} \\
CF_4 = 100\cdot(1+10)^3 & = 133.10 & \text{then discount with } r_{0,4} \\
CF_5 = 100\cdot(1+10)^4 & = 146.41 & \text{then discount with } r_{0,5} \\
\text{and so on forever} & &
\end{array}
$$

The PV of a growing perpetuity can be quickly computed as:

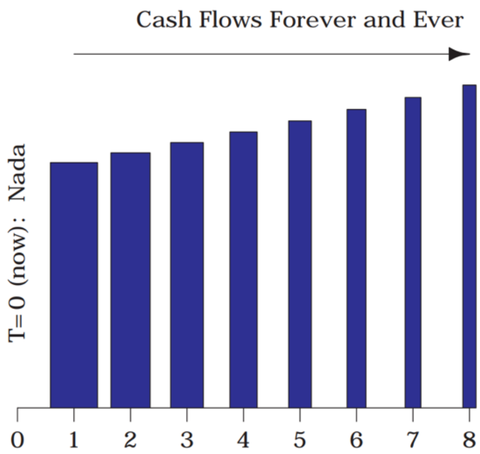
$$PV = \sum_{t=1}^{\infty} \frac{(CF_1 \times (1+g)^{t-1})}{(1+r)^t} = \frac{CF_1}{r-g}$$

You must memorize the RHS formula, and know what it means!

- The growth term acts like a reduction in the interest rate.
- The time subscript for the payment matters now, because $C_1 \neq C_2 \neq C_t$

How Can a Growing Infinite Sum Be Less Than ∞ ?

Because the growth is not fast enough. This is only the case if r > g. The formula makes no sense if r < g.



- What is the value of a promise to receive \$10 forever, beginning next year, if the interest rate is 5% per year? $PV_0 = C/r = \$10/5\% = \$200$

- What would be the value of that same promise but if you receive \$10 starting now and each year thereafter forever? What would that perpetuity formula be? $PV_0 = \$10 + C/r = \$210$

- What is the value of a firm that pays \$10 this year, growing by 2% forever, if interest rate is 5% per year? PV = \$10 + \$10.2/(5% - 2%) = \$350

- In 10 years, a firm will have annual cash flows of \$100 million. Thereafter, its cash flows will grow at the inflation rate of 3%. If the applicable interest rate is 8%, what is its value if you sell the firm in 10 years? What would this be worth today?
  $PV_{10} = C_{11}/(8\% - 3\%) + C_{10} = \$103/(8\% - 3\%) + \$100 = \$2.16$ billion
  $PV_0 = \$2.16$ billion$/(1 + 8\%)^{10} = \$1$ billion

Common Usage of Growing Perpetuity Formula
Growing perpetuity shortcuts are commonly used, and in many contexts. The most prominent use occurs in "pro-formas", where growing perpetuities are typically used to "guess-timate" the present value of the residual firm value after an arbitrary T years in the future:

- A common long-run growth rate in this formula is then often the inflation rate.
- The first T years are computed in detail.
- Typical T's in pro-formas are about 10 years.

The Gordon Dividend Growth Model
Using D for CF gives you the Gordon Dividend Growth Model.
Don't trust the GDGM: Dividends can be rearranged. In fact, there is a fairly strong irrelevance proposition here: Given its underlying projects, it should not matter whether the firm pays out $1 or $10 in dividends.

- Fewer dividends paid today will imply more dividends next year.
- Thus, expected rates of returns obtained from GDGM are suspect.

A better version, although without a fancy name, uses earnings instead of dividends. There can be a similar irrelevance proposition for earnings as there is for dividends [firms can move earnings across periods], but it is not as easy/legal to shift earnings.
GDGM also used to get implied cost of capital: r = g + D/P is the expected rate of return implied in today's stock price. A higher price today implies a lower implied cost of capital at which the firms can obtain capital from investors.

- What should be the share price of a firm that pays dividends of $1/year, whose dividends have grown by 4% every year and will continue to do so forever, if its cost of capital is 12% per annum? P = D/(r – g) = $1/(12% - 4%) = $12.50

- What is the cost of capital for a firm that pays a dividend yield of 5% per annum today, if its dividends are expected to grow at a rate of 3% per annum forever? D_1/P_0 = 5% and g = 3%, $P_0 = D_1/r - g$ → (r – g) = D/P → r = D/P + g = 5% + 3% = 8%

- In 2000, the P/E ratio of the stock market reached about 45. If you assume that these corporations will grow roughly at the overall economy's (GDP) growth rate of 4-5% per year, what should investors have reasonably expected in terms of a likely future rate of return implied by the stock market's level? P = E / (r – g) → P/E = 45 & g = 5% → P/E = 1 / (r – g) → r = g + 1 / (P/E) = 5% + 1/45 = 7.2%

Annuities
An annuity is a financial instrument that pays CF dollars for T years. Its PV formula is:

$$PV = \sum_{t=1}^{T} \frac{CF}{(1+r)^t} = \left(\frac{CF}{r}\right) \times [1 - \frac{1}{(1+r)^T}]$$

Again, make sure you know when the first cash flow begins. It starts tomorrow [t=1], not today [t=0]. You should remember this formula, or at least be able to quickly derive it. I remember the formula, because an annuity is a perpetuity today, minus a (properly discounted) perpetuity in the future:

$$PV = \left(\frac{CF}{r}\right) - \frac{1}{(1+r)^T} \times \left(\frac{CF}{r}\right)$$

Annuity Example - Mortgage Loan

Here is a summary of how mortgage payments are usually calculated:

- A 30-year mortgage is an annuity with 360 monthly payments, starting one month from today.
- Because payments are monthly, we need the monthly interest rate.
- The monthly rate on a mortgage is always computed as the quoted rate divided by 12 (in other words, like bank interest, your actual interest rate on a mortgage is higher than quoted. Lovely, isn't it?)
- So, the monthly interest rate on a 9% mortgage is: $r_{monthly} = \frac{0.09}{12} = 0.0075\ per\ month$

- To buy a house, you need to take out a $1,200,000 fixed rate mortgage with 30 years to maturity, 360 equal monthly payments, and a quoted interest rate of 9%. (You could also call this 9% compounded monthly.) What will be your monthly mortgage payment? PV = CF/r x [1 – 1/(1 + r)^T] where T = 360, PV =1,200,000, r_monthly = 9%/12 = 0.75%.
  $1,200,000 = CF/0.75 * [1 – 1/(1 + 0.75%)^360 → CF = ($1,200,000 x 0.75%)/[1 – 1/(1 + 0.75%)^360] = $9,655.47

- Assume the interest rate is r = 20%. You need to rent a building. The landlord gives you two choices, a two-year lease with an extra payment in the first year, or a three-year lease:

Time	0	1	2
Project A (or Rent A)	$20	$12	
Project B (or Rent B)	$15	$15	$15

A. Which is better?
B. What is the PV (or cost) of a project A?
   PV_A = $20 + $12/(1 + 20%) = $30 and PV_B = $15 + $15/(1 + 20%) + $15/(1 + 20%)^2 = $37.9
C. What is the "equivalent annual rent" of project A?
   EAR_A = C + C/1.2 = $30 then C = $30 * 1/(1 + 1/1.2) = $16.36 > $15
   Then we would choose B. And C is the Cash Flow

Level-Coupon Bonds

Most bonds are coupon bonds, i.e., they have interim payments. Most corporate bonds are level-coupon bonds, this means that the coupon level will remain the same over the life of the bond.
Most common bond: **x% semi-annual level coupon bond**.

- Take the principal (often $1,000 for corporate bonds), multiply it by x% to obtain the annual coupon payment, divide it by two, and this is the coupon that is paid every six months.
- For example, a 8% semi-annual level coupon bond pays $40 every six months on $1,000 in principal. At maturity, it pays $1,040.
- **The x% is not the interest rate implicit in the bond!** It is a standard way to tell you the payment flow of this bond.

The book works out a full example of level coupon pricing. Tedious but straight-forward. Make sure you can do it, too; we just cover some preliminaries here.

- What are the payments to 5% semi-annual level coupon bond, $100 million, due in 2.5 years?

- A zero bond has no interim payments. How do you earn interest on a bond that gives you no interest payments? The principal is discounted.

- Is the coupon rate of a bond equal to the interest rate? No

- What is the interest rate on an IBM 3.5% semi-annual coupon bond? We cannot compute the interest rate without the data regarding the principal.

- An insurance company offers a retirement annuity that pays $100,000 per year for 15 years and sells for $806,070. What is the implied interest rate that this insurance company is offering? $\$806,070 = \$100,000/r * [1 – (1/1 + r)^15]$, verify that $r = 9\%$.

- An insurance company offers a retirement annuity that pays $100,000 per year for 15 years, growing at an "inflation-compensator" rate of 3%, and sells for $806,070. What is the interest rate? $\$806,070 = (\$100,000/ (r – 3\%))[1 – (1 + 3\%/1 + r)^15]$, verify that $r = 11.75855\%$

- The prevailing interest rate is 10%/year. If you put aside $1,000,000 to cover 18 years of expenses, how much could you draw down each year? $\$1,000,000 = x/10\% * [1 -1/(1 + 10\%)^18] \rightarrow x = \$1,000,000/ [1 – 1/(1 + 10\%)^18]*10\% = \$121,930$

- The prevailing interest rate is 10%/year. If you want to draw $100,000 each year to cover 18 years of expenses, how much would you have to set aside? $PV = \$100,000/10\% * [1 – 1/(1 + 10\%)^18] = \$820,141$

- Assume that our firm has stopped growing in real terms, and the current interest rate is 6% per annum. The inflation rate is 2% per annum. This year, we earned $100,000. What is the value of the firm? (Do it over-the-envelope, and exact) [What is the first cash flow?]
  $C_1 = \$100,000 * (1 + 2\%) = \$102,000$
  $V_0 = C_1/(r-g) = \$102,000/(6\% - 2\%) = \$2,550,000$

- In 2016, GOOG (Google, called Alphabet today) had a P/E ratio of 27. Its cost of capital was 8%. If Alphabet lasts forever, what does the market believe its growth rate will be?
  $P = E/(r – g) \rightarrow P/E = 1/(r – g) \rightarrow (r – g) = 1/(P/E) \rightarrow g = r - 1/(P/E) = 8\% - 1/27 = 4.3\%$

- An example drawn from an actual automobile loan agreement: The advertisement claimed, "12 months car loans. Only 9%"
  A. For this 12-months $10,000 loan, at 9%, you owe $10,900. Thus, your monthly payments will come out to $10,900/12 = $908.33 per month. Ok? No
  B. What should be the PV of the car loan? $PV = \$908.33/0.721\% * [1 – 1/(1 + 0.721\%)^12] = \$10,406.04$

So, PV should be $10,406 but my loan is only $10,000. What does it mean?
  C. *Car deal gains $406*
  D. *The true r is not 9%*

- If you took out a loan from the bank at a true interest rate of 9% (8.649% quoted as compounded monthly), how much would the bank have asked you for the monthly payment?
  $(1 + 9\%)^(1/12) – 1 = 0.721\%$ monthly. So, bank quote = 0.721% * 12 = 8.649%
  Annuity formula: $\$10,000 = C/0.721\% * [1 – 1/(1 + 0.721\%)^12] = C * 11.46$.
  So, $C = \$10,000/11.46 = \$872.60$

- What is the actual interest rate (IRR) of the car dealer's deal? What is the cause of the difference? $\$10,000 = (\$908.33/1 + r_{month}) + (\$908.33/1 + r_{month})^2 + \ldots + (\$908.33/1 + r_{month})^12 = (\$908.33/1 + r_{month}) * [1 – 1/(1 + r_{month})^12]$. So, $r_{month} = 1.35\%$ and $r_{year} = (1 + 1.35\%)^12 - 1 = 17.459\%$

Extension – PV of Growing Annuity
The formula for a growing annuity.
A growing annuity pays $CF \times (1+g)^{t-2}$ per year starting in period 1 for T periods. Its present value is given by:

$$PV = \sum_{t=1}^{T} \left[\frac{(1+g)^{t-2}}{(1+r)^t}\right] \times CF = \left(\frac{CF}{r-g}\right) \times \left[\frac{(1+g)^T}{(1+r)^T}\right]$$

Example usage, you have an annuity that pays $100 in period 1. The annuity grows at a rate of 6% per year and pays off until and including period 10. The discount rate equals 8%.
Intuition of proof, it is a Growing Perpetuity minus a (properly discounted) Growing Perpetuity in the future.

Takeaways
Perpetuities and annuities are useful ways to compute PV under certain assumptions.
Growing perpetuity formula often used in financial valuation:

$$PV = \sum_{t=1}^{\infty} \left[\frac{CF_1 \times (1+g)^{t-1}}{(1+r)^t}\right] = \boldsymbol{\frac{CF_1}{r-g}}$$

Capital Budgeting

Maintained Assumptions
Today, we maintain the assumption of perfect markets, so we assume four market features:

- No differences in opinion
- No taxes
- No transaction costs
- No big sellers/buyers – infinitely many clones that can buy or sell

We again assume perfect certainty, so we know what the rates of return on every project are.
We again assume equal rates of returns in each period (year).

Definition of Capital Budgeting Rule
A **capital budgeting rule** is a method to decide which projects to take and which to reject.
Accept project if and only if NPV > $\mathbf{0}$ is the correct rule in a perfect market. Other rules can help your intuition at times, but they can only be either redundant or wrong. Some rules that are in common use, such as the payback rule, can be badly wrong and make less sense.

NPV – Why is NPV the Best Rule?
The reason is simple: if there were better rule that could come up with different answer in the simplest scenario (perfect markets, no uncertainty), it would leave good projects (money) on the table. This would be a mistake. You could "arbitrage" it.
Ergo, any alternative rule must converge to the NPV rule as the financial market gets closer to perfection – or this alternative rule is simply wrong.

How Common Should Positive-NPV Projects Be in a Perfect World?
In perfect markets under certainty, positive NPV projects are close to "arbitrage", money for nothing. (This extends to negative NPV projects if someone is willing to buy it from you).
Ergo, positive NPV projects should be hard to locate, unless you have some resources that are not widely available to everyone. What should happen in the real world if positive NPV projects were abundant is that the prevailing interest rate should adjust.

- You have $100 in cash. The prevailing interest rate is 20% per annum. You have two investment choices:
    A. Project costs $100 and will return $150 next year.
    B. Ice Cream – and you love ice cream.

  The problem is you know that you will be dead next year. What should you do? (Should you forego the ice cream for the greater social good and die unhappily?)

  Take the project: - $100.
  Sell the project to someone else: PV = $150/(1 + 20%) = $125.
  Buy $125 worth of ice cream.

- Does project value depend on when you need cash?

- In our perfect world, can you make your decision on investment and consumption choices separately, or do you need to make both of them at the same time (jointly)? *Yes, thanks to the separation theorem in perfect markets.*

- In a perfect market, how does project value depend on who you are (the identity of the owner)? *It doesn't.*

- Assume that we believe that the expected cash flow is $500 and the expected rate of return (cost of capital) is 20%. This is a 1-year project. Is it worse to commit an error in cash flows of in cost of capital?
  PV = $500/(1 + 20%) = $416.67.
  Suppose we make a 10% mistake in cash flow, then we have PV = $550/(1 + 20%) = $458.67. (**Bigger Mistake**).
  Suppose instead a 10% mistake in cost of capital, then PV = $500/(1 + 18%) = $423.73

- Does your conclusion above change if this is a 50-year project? We use the Perpetuity Approach: PV = $500/20% = $2,500
  Suppose we make a 10% mistake in cash flow, then we have PV = $550/20% = $2,750.
  Suppose instead a 10% mistake in cost of capital, then PV = $500/18% = $2,777.8. (**Bigger Mistake**).

Rate of Return with More Than One Inflow
What is the holding rate of return on a project that costs $13.16 million, and pays $7 million next year, followed by $8 million the year after?

Time	0	1	2
Cash Flow	−$13.16	$7	$8

IRR – The Internal Rate of Return
To answer the previous question, you need a measure that generalizes the rate of return to more than one inflow and one outflow. The most prominent such measure is the internal rate of return.

- The IRR (Internal Rate of Return) of a project is defined as the rate-of-return-like-number which sets the NPV equal to zero:
$$0 = C_0 + \frac{\mathbb{E}[C_1]}{(1+\mathrm{IRR})} + \frac{\mathbb{E}[C_2]}{(1+\mathrm{IRR})^2} + \frac{\mathbb{E}[C_3]}{(1+\mathrm{IRR})^3} + \ldots$$

  In the context of bonds, IRR is called Yield-To-Maturity (YTM).

Example: C_0 = \$-13.16, C_1 = \$7, C_2 = \$8. Solve:

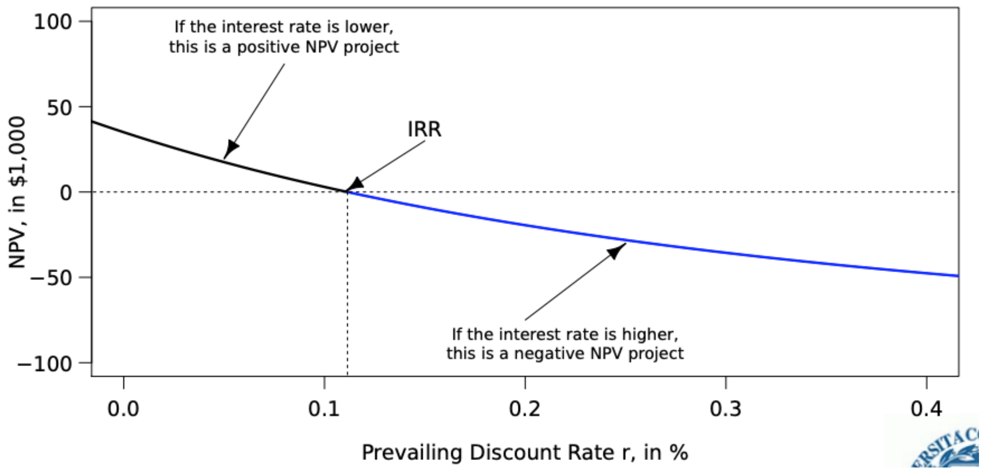
$$-\$13.16 + \frac{\$7}{(1+\text{IRR})} + \frac{\$8}{(1+\text{IRR})^2} = 0 \iff \text{IRR} \approx 9\%$$

If there is only one inflow and one outflow, then the IRR is the rate of return. IRR generalizes the rate of return.

IRR is in common use. You must understand it inside-out.

IRR and NPV

Project Flows: -\$100, \$5, \$10, \$120



At IRR = 12%, this is a 0-NPV project.

The Concept of IRR

The IRR is not a rate of return in the sense that we define a rate of return earlier as a holding rate of return, obtained from investing C_0 and later receiving C_t. If there are only two cash flows, IRR simplifies into the rate of return. IRR is a "characteristic" of a project's cash flows. It is purely a mapping from – i.e., a summary statistic of – many cash flows into one single number, just like the average cash flow or standard deviation of cash flow or auto-correlation of cash flows are:

- Intuitively, you can consider an "internal rate of return" to be sort of a "time-weighted average rate of return intrinsic to cash flows" – similar to a rate of return.
- Intuitively, a project with a higher IRR is more "profitable".
- Multiplying each and every cash flow by the same factor, positive or negative, will not change the IRR. IRR is **Scale-Invariant**.

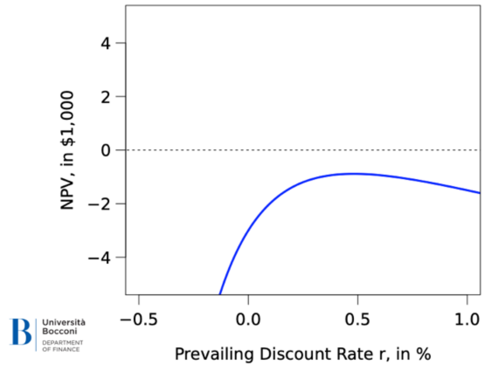
Finding the IRR

There is no general algebraic closed-form formula that solves the IRR for many cash flows. The IRR solution is the xero-point of a higher-order polynomial. With three or more cash flows, this a mess or impossible. Manual iteration = Intelligent trial-and-error.
Many spreadsheets and calculator have trial-and-error methods built-in. For example, in Excel, this function is called IRR(). You can find an example of how to use it in the book.

- If C_0 = \$40, C_1 = -\$80, C_2 = \$104, what is the IRR?
  When we have all cash flows of the same sign it does not exist an IRR.

No IRR

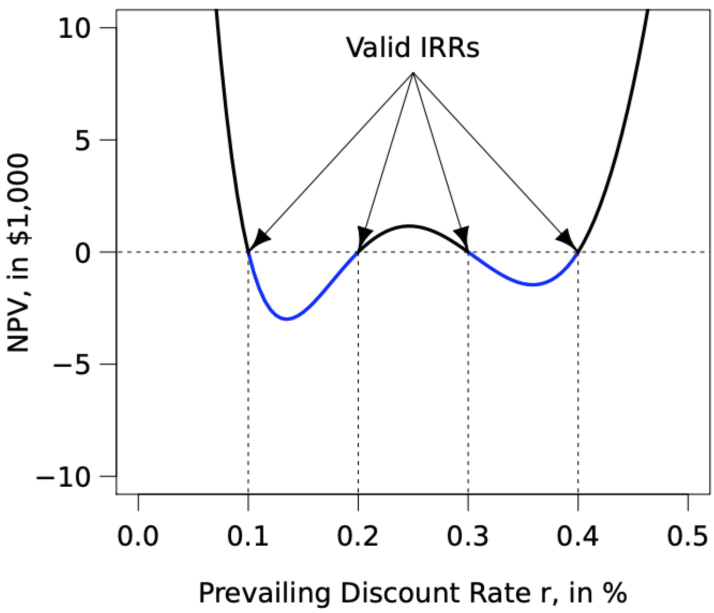
The project is positive or negative NPV for any interest rate.



- If C_0 = -\$100, C_1 = +\$3600, C_2 = -\$431, C_3 = +\$171.60 what is the IRR?
  - *A.* Is 10% the IRR? Yes
  - *B.* Is 20% the IRR? Yes
  - *C.* Is 30% the IRR? Yes
  - *D.* What answer will Excel give? It gives 10%, not so precise as an instrument to spot IRRs.

An Example of Multiple IRRs

These cutoffs define regions of IRR where you would or would not take the project. Rather than trying to compute those, it's better to use NPV.



Are These Irrelevant and Absurd IRR Problems?

A little but not greatly. You are guaranteed one unique IRR if you have at first only up-front cash flows that are investments (negative numbers), followed only by payback (positive cash flows) after the investment stage.

- This cash flow pattern is the usual case for financial bonds. Thus, the YTM for bonds is usually unique.
- This cash flow pattern is also usually the case for most normal corporate investment projects.

- In the real world, most projects do not have both positive and negative cash flows that alternate many times (**but** there are projects that require big overhauls/maintenance, in which case it can happen).

You need to be aware of these issues, lest they bite you one day…

- PS: You will soon learn the difference between promised and expected returns. An IRR based on promised cash flows is a promised IRR. It should never be used for capital budgeting purposes (for useful IRR [capital budgeting] calculations, you need to use expected cash flows in the numerator, not the promised ones – just as you need to do in the NPV context).

IRR as a Capital Budgeting Rule

The IRR capital budgeting rule is:

- If the project begins with only money out, followed by only money in:
  Invest If: Project IRR > Cost of Capital (r).

- If the loan begins with only money in, followed by only money out:
  Borrow If: Project IRR < Cost of Capital (r).

In case of sign doubts, calculate the NPV!

Notes

- The IRR rule leads often (but not always) to the same answer as the NPV rule, and thus the correct answer. This is why IRR has survived as a common method for "capital budgeting".
- Because you cannot improve on "correct", the **NPV is at least as good as the IRR** as a capital budgeting rule.
- If you use IRR correctly, it can give you nice extra intuition.
- IRR's Advantage: You can use it before you know your cost of capital.
- IRR uniqueness and multiplicity problems apply in this context, too.

- The prevailing cost of capital is 20%. Now, consider two mutually exclusive projects:
  A. A: C_0 = -\$80, C_1 = +\$50, C_2 = +\$100
  B. B: C_0 = -\$85, C_1 = +\$100, C_2 = +\$45

  C. Which one should you take?
     A: NPV_A = -80 + 50/(1 + 20%) + 100/(1 + 20%)^2 = \$31.1
     B: NPV_B = -85 + 100/(1 + 20%) + 45/(1 + 20%)^2 = \$29.58
     Since they are mutually exclusive and \$31.1 > \$29.58, then we'd better take A.

  D. Are 42%, 47%, 52%, or 57% the projects IRRs of A and B?
     IRR_A: r_A such that: NPV_A = 0, -80 + 50/(1 + r_A) + 100/(1 + r_A)^2 = 0 and r_A = 47%
     IRR_B: r_B such that: NPV_B = 0, -85 + 100/(1 + r_B) + 45/(1 + r_B)^2 = 0 and r_B = 52%

  E. If you can take only one of the two projects, which is better?
     According to NPV we know we should take A. However, the IRR is scale invariant and the r is larger in B, but we should focus on NPV.

IRR vs NPV Capital Budgeting Rules

Disadvantages of IRR:

- IRR is scale insensitive (which causes problems comparing projects).
- There may be no IRR.
- There may be multiple IRRs.

- The benchmark cost of capital may be time-varying, in which case the IRR capital budgeting rule fails.

Advantages of IRR:
- Your cost of capital (the prevailing r) does not enter into the IRR calculation.
- You do not need to recalculate the whole project value under different cost-of-capital scenarios (if you want to play around with projects before talking to investors).

Other Rules - The Profitability Index

Time	0	1	2
Cash Flow	–\$13.16	+\$7	+\$8

Used occasionally. Not as common as IRR.
The profitability index is the PV of future cash flows, divided by the cost (made positive). Here, if r = 20%, then:

$$\text{PI} = \frac{PV(\$7, \$8; 20\%)}{-(-\$13.16)} = \frac{\$11.39}{\$13.16} \approx 0.8655$$

If r = 5%, then:

$$\text{PI} = \frac{PV(\$7, \$8; 5\%)}{-(-\$13.16)} = \frac{\$13.92}{\$13.16} \approx 1.058$$

The Profitability Index as a Capital Budgeting Rule
Capital Budgeting Rule:
Invest if PI > 1. Reject if PI < 1.

Often gives the same recommendation as NPV.
Shares all the same problems as IRR. Most importantly, it lacks the concept of project scale, which is a problem for "either-or" projects (higher PI projects are not necessarily better than lower PI projects).
Does not have the main advantage of IRR (which is that the cost of capital is kept separate).

Other Investment Rules – Payback
The most common alternative rule is the so-called "payback rule". It measures how long it takes to get your money back.
Capital budgeting rule version: take projects with shortest payback time.
Which project is better?

	Time	0	1	2	Payback
A	Cash Flow	-\$1	+\$2		
B	Cash Flow	-\$1		+\$200	

According to Payback Rule is better project A, but is clearly idiotic.

Advantage of Payback
It may be useful if managers cannot be trusted to provide good estimates of far out future cash flows.
It's harder to lie if you have to claim that you can prove project profitability within 1 year.
In a perfect market, you know what these cash flows are. So, trusting managers is irrelevant.
- If used for project accept/reject decision, many other rules are pretty dumb. They become a little bit smarter if you use them only for useful background information. It can also help in informal conversation, if capital is highly constrained, and/or if financial markets are not at

all perfect for you (Even in this case, a form of NPV with a higher discount rate may be better, though).
- When the other rules are stark, they may make the point that the NPV is very high in an intuitive and forceful manner. For example, a project that delivers all the money back by tomorrow is likely to have a pretty good NPV.

Real-Life Capital Budgeting Rules

Method	CFO Usage	Yields Correct Answer	Main Explanation
Internal Rate of Return (IRR)	(76%)	Often	Chapter 4
Net Present Value (NPV)	(75%)	(Almost) Always	Chapter 2
Payback Period	(57%)	Rarely	Chapter 4
Earning Multiples (P/E Ratios)	(39%)	With Caution	Chapter 15
Discounted Payback	(30%)	Rarely	Chapter 4
Accounting Rate of Return	(20%)	Rarely	Chapter 15
Profitability Index	(12%)	Often	Chapter 4

Note, "Rarely" means "usually no – often used incorrectly in the real world". NPV works if correctly applied, which is why I added the qualifier "almost" to always. Of course, if you are considering an extremely good or an extremely bad project, almost any evaluation criterion is likely to give you the same recommendation (Even a stopped clock gives you the right answer twice a day).

Takeaways
- Capital Budgeting Rules
  A. NPV always correct → should use NPV.
  B. IRR some time same as NPV → in such cases can also use IRR.
  C. Other rules can have serious pitfalls.

Yield Curve
Maintained Assumptions
Today, we maintain the assumption of perfect markets, so we assume four market features:
- No differences in opinion
- No taxes
- No transaction costs
- No big sellers/buyers – infinitely many clones that can buy or sell

We again assume perfect certainty, so we know what the rates of return on every project are.
But we no longer assume equal rates of return in each period (year)!
Oranges cost more in the winter than in the summer. So why can't project payoffs not have different prices (rates of return) if they will realize at different times?

Time-Varying Returns
All earlier formulas hold, only difference: $(1 + r_{0,t}) \neq (1 + r)^t$
Main complication, we now need many subscripts. For example:

$$(1 + r_{0,3}) = (1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot (1 + r_{2,3})$$

$$NPV = C_0 + \frac{C_1}{(1 + r_{0,1})} + \frac{C_2}{(1 + r_{0,2})} + \frac{C_3}{(1 + r_{0,3})}$$

$$= C_0 + \frac{C_1}{(1 + r_{0,1})} + \frac{C_2}{(1 + r_{0,1}) \cdot (1 + r_{1,2})} + \frac{C_3}{(1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot (1 + r_{2,3})}$$

Time-Varying Rates of Returns More Formally

If you like it more formal,
$$(1 + r_{t,t+i}) = (1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}) \cdots (1 + r_{t+i-1,t+i})$$
$$= (1 + r_{t+1}) \cdot (1 + r_{t+2}) \cdots (1 + r_{t+i}) = \prod_{j=t+1}^{t+i} (1 + r_j)$$
$$PV = \sum_{t=1}^{\infty} \left(\frac{CF_t}{(1 + r_{0,t})} \right) = \sum_{t=1}^{\infty} \left[\frac{CF_t}{\prod_{j=1}^{t} (1 + r_j)} \right]$$

Recall that r_j is an abbreviation for $r_{j-1,j}$.

- Your project will give you a rate of return of 100% (double your money) over 15 years. Is this a lot or a little?
  (1 + 100%) = (1 + x)^15 → x = (1 + 100%)^(1/15) – 1 = 4.73%

- How does this 15-year rate of capital accumulation compare to a rate of capital accumulation of 1% over 3 months?
  (1 + 1%)^4 = 1 + y → y = 4.06% < 4.73%. We can compare these two rates because they are both annual rates of return. *(To the power of 4 because we have 4 quarters of 3 months)*.

- If the 1-year interest in year 1 is 5%, and the 1-year interest rate in year 2 will be 3%, what is the annualized interest rate? Annualized Interest Rate = 3.995%

Interpreting Annualized Interest Rates

Is an annualized interest rate more like an average or more like a sum?

- It is more like an average. Indeed, it is called the geometric average!

Important Remark

Almost all interest rates are quoted in annualized terms.

Annualized interest rates are (often just a little) below average interest rates, because they take (away/into account) the interest on interest.

Inflation

Inflation – Real and Nominal Rates

- A nominal cash flow is simply the nominal number of dollars you pay out or receive. A real cash flow is adjusted for inflation.
- A real dollar (/euro) always has the same purchasing power.
- If the EU were to call everything that is a cent today a euro henceforth, instant inflation would be 9,900% - and yet it would not matter as long as all contracts today are clear about the units (euros) and their transactions.

If properly contracted for, inflation is not a market imperfection.

- Just because quoted prices are less in Euros than in Lira can be called inflation, but it does not in itself create a problem. As a further clarification, just realize that a Euro is not the same as a Lira. In the same way, a Euro next year is not the same as a Euro this year.
- In sum, inflation per se is not a friction (or market imperfection) – if everything is contracted in real terms. However, in the real world, most contracts are in nominal terms, so as an investor you must worry about inflation, and sometimes it can have similar effects to those of a market imperfection.

An Example with Cash Flow and Inflation

- You have $100, which you invest for 1 year at 10%.

- One loaf of bread sells for $2.00 today.
- Your $100 can purchase 50 loaves today.
- Bread Inflation over the next year will be 4%. How is inflation (the CPI) defined?
- The bank pays a nominal rate of return of 10% per year.

- What is your Real Rate of Return?
  *Next year, the Bank will pay you $110 (Nominal Dollars). Next year, one loaf of bread will cost $2 * (1 + 4%) = $2.08. If you put your money in the bank, next year you will purchase $110/$2.08 = 52.88 loafs of bread. Therefore, the Real Rate of Return is (52.88/50 – 1) = 5.77%. So, with an inflation rate of 4% and a nominal real rate of return of 10% , you will face a real rate of return of 5.77%*

- What is the formula that relates the nominal rate, the real rate, and the inflation rate?
  (1 + Real Rate) = (1 + Nominal Rate) / (1 + Inflation Rate)

More About the Inflation Adjustment Formula
More generally: **(1 + Real Rate) * (1 + Inflation Rate) = (1 + Nominal Rate).**
Intuition, why is this a "one-plus" type formula? When all rates are very small, the approximation:
Real Rate + Inflation Rate ~ Nominal Rate

Could be acceptable, **depending on the circumstances**, but this approximation formula is **not exactly correct**. Why? Consider this:

- One real dollar today equals one nominal dollar today (usually!)
- An inflation-adjusted dollar is $1/(1 + π). So, $110 next year is $100/1.04 = $105.77 today in inflation-adjusted dollars. $100 nominal next year is $96.15 real dollars today.
- Sometimes, real dollars are also called "inflation-adjusted" dollars, or – and this is where it gets awful – are even called "in today's dollars". Unfortunately, different people mean different things by these phrases.

Important Remark on Real and Nominal Discounting
- Either discount nominal dollars with nominal interest rates, or discount real dollars with real interest rates.
- Never mix nominal cash flows with real rates, or vice versa.

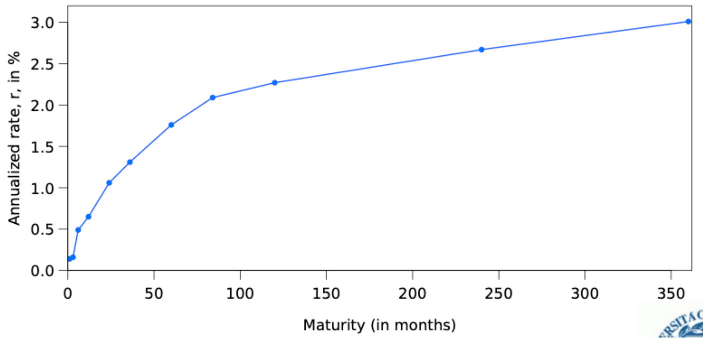
Yield Curve
The Yield Curve and US Treasuries
US Treasuries are the most important financial security
- The outstanding amount was almost $18 trillion in 2015.
- Annual trading is around $100-$150 trillion. (Turnover = 5-10 Times!)
- Names: Bills (-0.99yr), Notes (1yr – 10yr), Bonds (10yr -).
- Only the mortgage bond market is bigger than the UST market.

This market is close to "perfect":
- Extremely low transaction costs (for traders).
- Few opinion differences (inside information).
- Deep market – many buyers and sellers.
- Income taxes depend on owner.

In addition, there is (almost) no uncertainty about repayment (PS: a market could still be perfect, even if payoffs are uncertain).

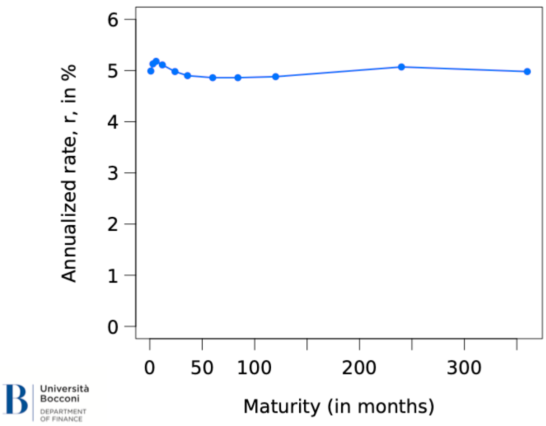
(Zero-coupon) US Treasuries are among the simplest possible financial instrument in the world. **Definition**, The Yield Curve is the plot of annualized yields (Y-axis) against time-to-maturity (X-axis).

Yield Curve December 2015

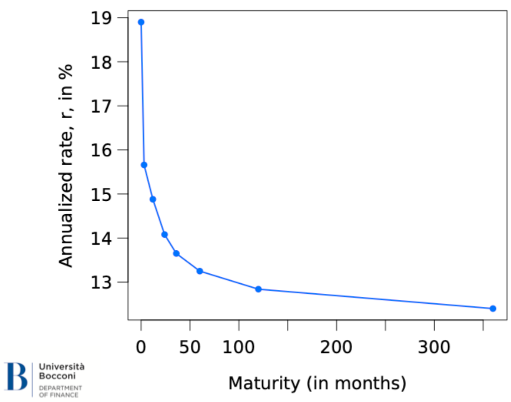


- Can the yield curve be flat? Yes
- Can the yield curve slope down? Yes

Yield Curve January 2007



Yield Curve December 1980



Time – Varying Cost of Capital

The yield curve is a fundamental tool of finance. It always graphs annualized rates. It measures differences in the costs of capital for (risk-free) projects with different horizons.

In the real world, many variations on the yield curve are in use, e.g., yield-curves constructed from risky corporate bonds or yield-curves constructed from foreign bonds.
If not further qualified, when someone talks about the yield curve, they mean the yield curve on US Treasuries.

Names: Spot and Forward Rates
We call a currently prevailing interest rate for an investment starting today a *spot* interest rate.
A forward rate is an interest rate that will begin with a cash flow in the future. It is the opposite of a spot rate. Like all other interest rates, spot and forward rates are usually quoted in annualized terms.

We denote an annualized interest rate over 15 years as $r_{\overline{15}}$. This contrasts with our notation for the 15-year non-annualized holding interest rate, which is $r_{0,15}$.

$$(1 + r_{\overline{15}})^{15} \equiv (1 + r_{0,15}) \equiv (1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot \ldots \cdot (1 + r_{14,15})$$

$$\iff (1 + r_{\overline{15}}) \equiv (1 + r_{0,15})^{1/15}$$

$$(1 + r_{\overline{t}})^t \equiv (1 + r_{0,t})$$

$$\iff (1 + r_{\overline{t}}) \equiv (1 + r_{0,t})^{1/t}$$

$$\textit{Example:}\ r_{0,5} = 27.63\% \iff r_{\overline{5}} = 5\% \ .$$

This is the notation used in our course, and not necessarily used elsewhere. To make matter worse, some people will use R to mean 1 + r, believing you can figure out whatever they may have meant. Others will just capitalize R and mean the same thing, namely r.

Notation Summary

$$(1 + r_{0,1}) = (1 + r_{\overline{1}})^1 = (1 + r_{0,1})$$

$$(1 + r_{0,2}) = (1 + r_{\overline{2}})^2 = (1 + r_{0,1}) \cdot (1 + r_{1,2})$$

$$(1 + r_{0,3}) = (1 + r_{\overline{3}})^3 = (1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot (1 + r_{2,3})$$

The interest rate from period 1 to period 2 is called the 1-Year Forward (Interest) Rate from Period 1 to Period 2. In a world of certainty, the forward rate will be the future spot rate: We know it!
Later I will show you how you can make the forward rate your personal future spot rate, even in a world of uncertainty.

Two Useful Approximate Answers
Remember:

- An annualized rate of return is "more like the average"
- A holding rate of return is "more like the sum"

An Exercise

A 1-year bond has an (annual) rate of return of 5%. When the first bond will come due, you will be able to purchase another 1-year bond that will have an (annual) rate of return of 10%. When the second bond will come due, you will be able to purchase another 1-year bond that will have an (annual) rate of return of 15%.
What are the three total holding rates of returns, and the three annualized rates of return?
(Calculator FORBIDDEN. Use your intuition!)

Rates of Return

Spot + Forward	Holding	Annualized
$r_{0,1} = 5\%$	$r_{0,1} =$	$r_{\overline{1}} =$
$r_{1,2} = 10\%$	$r_{0,2} =$	$r_{\overline{2}} =$
$r_{2,3} = 15\%$	$r_{0,3} =$	$r_{\overline{3}} =$

- What are the three holding rates exactly?
- What are the three annualized interest rates exactly?

$r_{0,1}$: Spot + Forward = 5%, Holding = 5%, Annualized = 5%

$r_{1,2}$: Spot + Forward = 10%, Holding = (1 + 5%)*(1 + 10%) - 1 = 15.5%, Annualized = sqrt(1 + 15.5%) – 1 = 7.471%

$r_{2,3}$: Spot + Forward = 15%, Holding = (1 + 5%)*(1 + 10%)*(1 + 15%) - 1 = (1 + 15.5%)*(1 + 15%) – 1 = 32.825%, Annualized = sqrt^3(1 + 32.825%) – 1 = 9.92%

- A 1-year bond has an annualized rate of return of 5% per year. A 2-year bond has an annualized rate of return of 10% per year. A 3-year bond has an annualized rate of return of 15% per year. First without calculator, then with: what are the three (total) holding rates of return?
  1-year Holding Rate = 5%,
  2-year Holding Rate = (1 + 10%)^2 – 1 = 21%,
  3-year Holding Rate = (1 + 15%)^3 – 1 = 52.0875%

- First without calculator, then with: what are the three annual rates of return?
  $r_{0,1}$ = 5%,
  $r_{1,2}$ = (1 + 21%)/(1 + 5%) – 1 = 15.2%,
  $r_{2,3}$ = (1 + 52.0875%)/[(1 + 5%)(1 + 15.2%)] – 1 = 0.25734

Summary

Rates of Return

Spot + Forward	Holding	Annualized
$r_{0,1} =$	$r_{0,1} =$	$r_{\overline{1}} =$
$r_{1,2} =$	$r_{0,2} =$	$r_{\overline{2}} =$
$r_{2,3} =$	$r_{0,3} =$	$r_{\overline{3}} =$

When you work with the yield curve, use over-the-envelope intuition to know the order of magnitude of your answers before you calculate them.
Because the annualized yield is an average of spot/forward rates, the forward rates rises/declines faster than the yield curve. For example, if r_1 = 5% and r_2 = 6%, then $r_{1,2}$ > 6%, because 5% and $r_{1,2}$ "geo-averaged" must come to 6%. By this argument, $r_{1,2}$ should be about 7%.

- Given a full set of annualized interest rates, r_1, r_2, …, r_T, can you compute each and every T-year holding rates of return? **Yes**

- Given a full set of annualized interest rates, r_1, r_2, …, r_T, can you compute each and every forward rate (implied future spot rate) from T – x years to T years? **Yes**

- Does the "yield curve" imply a unique set of forward/spot rates? **Yes**

- Does the complete set of spot holding rates of returns imply a unique yield curve? **Yes**

- Does the complete set of spot prices on zero (or other) bonds imply a unique yield curve? **Yes**

Summary
The "yield curve" or "term structure of interest rates" is the curve plotting the spot (i.e., annualized) interest rate on the y-axis against the time of the payment on the x-axis. It implies all forward interest rates.

What does an Upward – Sloping Yield Curve mean for an Investor?

- Higher future inflation? (Not usually)
- Higher future interest rates? (Not usually)
- Bargains? (Not usually) **Next**
- Risk Compensations? (most likely, yes) **Next**
  In the real world you have a choice: Lock in the future interest rates (which gives you what we calculated). These financial markets are close-to-perfect, so there is very little transaction cost to do this. Or take your chances: future actual interest rates may be higher or lower that the interest rates you could lock in today. A "risk premium" in which risk is higher for longer-term investment (e.g., if the firm can go bankrupt or inflation may erode the value of the repayment) would imply an upward-sloping yield curve.

Bond Price Sensitivity and Maturity
What happens to the value of a bond (or a loan) that you already own when interest rates increase? Does loan length (maturity) matter?

- Here is an example of a bond promising 8%/year:
  - A 30 year bond that promises 8% interest rate costs ($\$100/1.08^{30} \approx$) \$9.94 for each \$100 promise in payment
  - If the interest rate increases by 10 basis points, the price changes to \$9.67
  - The holding rate of return is $\$9.67/\$9.94 - 1 \approx -2.74\%$. For each \$100 in investment, you would have just lost \$2.74!
  - For a 1-year bond, the same calculation $p_0 = \$100/1.08 \approx \$92.5926$, $p_1 = \$100/1.081 \approx \$92.507$, and $r = p_1/p_0 - 1 \approx -0.09\%$
  - For a 1-day bond, the calculation $p_0 = \$100/1.08^{1/365} \approx \$99.979$, $p_1 = \$100/1.081^{1/365} \approx \$99.9787$, and $r = p_1/p_0 - 1 \approx -0.00025\%$. In fact, a 1-day bond is practically risk-free

Conclusion, the interest rate sensitivity of a 30-year bond is much higher than that of a 1-year (or 1-day bond).

Is A 1-Day Bond Riskier Or A 30-Year Bond?
If 10bp interest rate changes are equally likely for the economy-wide 30-year rate as they are for the economy-wide 1-day rate, then 30-year bonds are riskier investments. In the real world, short-rates changes of 10bp are more common for short (1-year) economy-wide rates than for long (30-year) economy-wide rates, but they are not so common as to negate the fact that the 30-year is riskier than the 1-year.

If we allow for uncertainty, long-term bond investors can get higher return for two reasons:

- Because of higher expected rates of returns in the future [e.g., due to higher future inflation rates], or
- Because they are earning a "risk premium".

The empirical historical evidence suggests that most of the time, the upward slope was more of a risk premium than an expectation of higher future rates.

Corporate Lessons of Non-Flat Yield Curves

A project of x-years is not simply the same as investing in x consecutive 1-year projects. From an investment perspective, they are different animals, and can require different costs of capital. The fact that longer-term projects may have to offer higher rates of return (could but) need not be due to higher risk. Even default-free Treasury bond projects in the economy that are longer-term have to offer higher rates of return than default-free Treasury bond projects in the economy that are shorter term. (Of course, in the real world long-term projects **are also often riskier** [default more often], and this also helps explain why long-term projects have to offer higher rates of return).

Duration

A project that pays $200 in one year and $100 in five years has a maturity of five years, the same as a "zero-" project that pays only $300 in five years. However, the first project is clearly shorter-term. Duration is a measure of when the cash flow arrives "on average". It is in common use in the bond context, but useful for all sorts of projects.

$$\text{Duration} = \frac{1 \times \$200 + 2 \times \$0 + 3 \times \$0 + 4 \times \$0 + 5 \times \$100}{\$200 + \$0 + \$0 + \$0 + \$100} \approx 2.3$$

A common variant called Macaulay duration uses the present value and not raw cash flows. It tilts more towards the front (i.e., this duration is smaller).

Takeaways

- Unequal rates of returns in each period → Yield Curve.
- Real and Nominal Rates: (1 + Real Rate) * (1 + Inflation Rate) = (1 + Nominal Rate).

Appendices

Looking Forward Rates: Given the current yield curve, you can lock in the future interest rate today. That is, you can eliminate all uncertainty about what interest rate that you will have to pay (or that you can earn). For example, you can buy and short Treasuries to lock in a 1-year saving Treasury rate for $1 million beginning in year 3 and lasting until year 4.

Future Interest Rates vs. Forward Rates: In the real world, future interest rates can be different from forward rates. Indeed, if you lock in a, say, 10-year-ahead 1-year savings interest rate today, on average you would have earned a higher rate of return than you would have if you had purchased 1-year savings bonds in the open market. If you are dealing with bonds, you therefore may need more notation. You now will have a future 1-year spot rate in 2030 (say $r_{2030,2031}$), and a 1-year forward rate that you can lock in today (say $f_{Now,2030,2031}$, which is the 1-year forward rate locked in today). Tomorrow's locked in forward rate would be $f_{tomorrow,2030,2031}$. And so on. Yikes.

Continuous Compounding: If interest is paid not once per year, but every second, this is the continuously compounded interest rate. It is often used for options pricing. OK, skipped for exams.

Strips: I cheated on the exact method to compute bond prices. The common yield curve is computed from IRRs, and not even based on actual interest rates, but based on interest quotes.

Uncertainty, Default and Risk

Maintained Assumptions

In this part (consisting of three chapters), we maintain the assumptions of the previous chapter:

We assume perfect markets, so we assume four market features:

- No differences in opinion
- No taxes
- No transaction costs
- No big sellers/buyers – infinitely many clones that can buy or sell

We already allow for different rates of returns in each period.
But now, we allow uncertainty. So, we do not know in advance what the rates of return on every project are. We need to predict (describe) the future. For this, we need statistics.

Warning: This Chapter May Be Illegal in Some States
Attend the rest of this course at your own risk:
"Persons pretending to forecast the future shall be considered disorderly under subdivision 3, section 901 of the criminal code and liable to a fine of $250 and/or six months in prison". – Section 889, New York State Code of Criminal Procedure.

Statistics
A **random variable** (denoted formally by a tilde over a variable in a statistics course) is not an ordinary variable. A random variable can take on a whole range of possibilities, which can be drawn in a histogram. In a sense, a random variable is more like a function of a randomizing device (e.g., a coin): it needs to obtain a value from this randomizing device. It may be useful to think of a random variable as being the histogram itself.

In most statistical applications, users usually *assume* that they know the histogram, but not the draw (outcome). This is great for a coin, die, or roulette. It sucks for the stock market, where we do not understand the underlying physics. On occasion, I will warn you about this *big leap* of faith.

Random Variable – Example
Our main random variable example will be the payoff you get after a die is thrown:
"1" = -$6; "2" = $36; "3" = -$12; "4" to "6" = $150.
Let's call this variable D.

The Expected Value of a Random Variable
E(.) is the abbreviation. For example, E(D). Think of the expected value (mean) of a random variable as the average if you repeat the experiment infinitely many times. From a random variable's histogram, you can calculate the expected value by multiplying each outcome by its probability, and adding them up.

- What is the Expected Payoff of our Random Variable D? 78$

- Is E(D) a random variable, too? No

- Is the expected payoff always the most likely outcome? No

- What is the expected value of the die-squared, E(Dˆ2)? $$11,496

- Is E[(Dˆ2)] the same as [E(D)]ˆ2? No

- What is a fair bet? What would it take for the above die-bet example to become a fair bet?
  A cost of $78 = Expected Payoff

Variance
The variance is Var(D) = E{[D – E(D)]ˆ2}, roughly speaking, the *expected squared deviation from the mean*. Think of the true unknown variance of a random variable as the computed squared deviation from the mean if you draw infinitely many realizations. In your statics class, you may also have encountered the "variance of the mean" estimate. This is a related concept used to test how likely it is that the underlying mean is actually zero. However, it is not the variance of a random variable.

Variance In Our Example
From the Random Variable's histogram, to obtain the variance, multiply each squared deviation from the mean by its probability, and add 'em up.

State	Prob	Outcome	Outcome − Mean	Squared
"1"	1/6	–$6	–$84	$$7,056
"2"	1/6	$36	–$42	$$1,764
"3"	1/6	–$12	–$90	$$8,100
"4" - "6"	3/6	$150	+$72	$$5,184
Weighted Mean:		$78	$0	$$5,412

The units on a variance are usually meaningless.

Standard Deviation
The standard deviation is the square-root of the variance: SD = sqrt($5,412) = $73.57
Intuitively, think of the standard-deviation is the "typical deviation from the mean" of the next draw. This is not entirely correct, but it is close enough and often helps the intuition.
If a variance is higher, then the standard deviation is higher. When people talk about risk, they often use phrasing such as "if the variance is higher". They could equally well say "if the standard deviation is higher". In fact, they often mean the latter. This is economics in action. People are mouth-lazy. Variance has 2 syllables, standard deviation has 5 syllables.

Big Leap of Faith – What is the True Histogram?
Assuming that we know the histogram is good for a throw of a die, where we know the physics. We do not really know the histogram for the rate of return on the stock market. Therefore, we pretend that the many historical outcome realizations of rates of returns can proxy for the true unknown histogram of rates of returns, and then we pretend that this histogram applies to future rates of return, too. This translation of the historical outcome histogram (distribution) into the future outcome histogram (distribution) is a heroic assumption – but it is often the only reasonable information that you have. You often have no better alternatives. If we use the historical realized distribution, some statistics require a small correction.

Extrapolating For The Future
Extrapolation works reasonably well for (short-term) standard deviations, but almost always poorly for means. Just because Google had an average 50% rate of return per year in the past does not mean it will have an expected rate of return of 50% per year in the future. For example, what is the expected rate of return on the stock market? Is it the same that it was historically? You must be mindful of where histo-extrapolation works and where it fails.
If you assume you know the histogram, then you can calculate the expectation.

- What is the expected mean and standard deviation if stock returns followed this historical distribution: +10%, -5%, +20%, +15%.
  E[R] = 0.25(10%) + 0.25(-5%) + 0.25(20%) + 0.25(15%) = 10%

R – E[R] = 0%, -15%, 10%, 5%
{R – E[R]}ˆ2 = 0 %%, 225%%, 100%%, 25%% → 350%%
Population: Var[R] = 350%%/4 = 87.5%% and SD = 9.35%
Sample: Var[R] = 350%%/3 = 116.67%% and SD = 10.8%

Estimating Stock Returns – Recap
We estimate a distribution of future rates of return from historical rate of return realizations. We assume each historical realization was an equally likely draw. When we compute the variance from a histogram that relies on a sample (not the known histogram of the population), we adjust by dividing not by N, but N – 1.
Wy Excel and Stats packages sometimes give different answers for Var and SD:

- We divide by N, because we presume you know the population. Excel and stats packages without the 'p' at the tail end of the function name divide by N – 1, because they presume you know only a sample drawn from the population. Usually, this makes little difference if you have reasonably large datasets – but it makes a difference in our small N = 4 datasets. So, you must use stdevp and varp, and not stdev and var to get the same results as those we compute in these slides in class.

Let me warn you again: Do not trust the historical means blindly especially when it comes to predicting future expected rates of returns. For individual stocks (rather than big diversified portfolios), this would be incredibly bad. Even for big diversified portfolios, this is a big leap of faith. In contrast, (recent) historical variances (and covariances and standard deviations) are usually better predictors of (short-term) future variances (and covariances and standard deviations). Advice: use a couple of years, say 1 – 5, of historical data to estimate them. However, a short time-series of historical numbers is usually not reliable enough to calculate tail-risk – the probability of a complete blow-up. How do you estimate the risk of the next Space-X rocket exploding?

Preferences

- When is risk neutrality or low risk-aversion a good assumption?
- Why do people smoke, climb mountains, drive motorcycles, play the lottery?

Credit Risk
Default ("Credit") Risk
Most loans have credit risk, in that the borrower can default. Loosely, it is the part of the return that is promised but will be lost on average because the borrower goes belly-up.

- Can the U.S. government default? Do Treasury securities have any default (credit) risk? Yes & No

Henceforth, assume that a U.S. government bond costing $200 promises a 5% interest rate, i.e., $210.

- Assume you are risk-neutral.
- I want to borrow $200 from you. I promise to repay $210.
- However, I may go bankrupt in 1 out of 100 cases, in which case I can repay only $50.

- What is your promised rate of return on my personal bond? 210/200 – 1 = 5%

- Do I promise to give you the same rate of return as the Treasury? Yes

- What would you expect my personal bond to return?
  Expected Payoff = 99% * $210 + 1% * $50 = $208.40
  Expected Rate of Return = 5%

- If you extend this loan to me, what rate of return would you expect my bond to give you?

- Would you prefer to make this loan or to put your money into the 5% government bond? U.S. Treasury Bond.

- How much money would you give me in exchange for my promise to pay you $210?
  99% * ($210/P0 - 1) + 1% * ($50/P0 – 1) >= 5%
  P0 = $208.40/1.05 = $198.5

- If the WSJ were to print my bond's interest rate, what interest rate would it print?
  Promised Rate of Return = $210/$198.5 – 1 = 5.81%

- Are (WSJ) quoted interest rates on risky bonds expected rates?

- In the real world, would this interest rate really be high enough?

Default (Credit) and Risk Premia
Default Premium vs Risk Premium.
- The default premium is compensation to make you break even. It is required to get you to participate even if you are risk-neutral.
- If you repeat the investment infinitely many times, the average default payment is 0. You get a positive amount if everything goes well, and a negative amount if default occurs, so you come out even.
- The risk premium is extra compensation above the time premium; it is only required to get you to participate if you are risk-averse.
- If you repeat the investment infinitely many times, the risk premium will allow you to earn more than an investor holding Treasuries.

Never confuse the credit premium with the risk premium. In our world, with a Treasury rate of 5% and a quoted bond of 5.81%, the risk premium was still zero.
Warning: You must be clear about the distinction between default premia and risk premia. Make sure you know what they are about, and know the difference between these concepts!

Generalization of Premium Decomposition
In a risk-neutral world:
- **Quoted (=Promised) Interest Rate > Expected Interest Rate**
- **Quoted Interest Rate = Time Premium + Default Premium**
- **Expected Interest Rate = Time Premium**

We are ignoring other premia for now, such as risk premia, liquidity premia, tax premia, contract premia, etc.
In our example, the promised interest rate was 5.81%, the time premium is 5%, the default premium was 0.81% and the risk premium is 0%.
Risk Premia for "fairly safe" bonds from "large, safe" companies are not too high; it is the default premia that jacks up the quoted interest rate.
IRR computed from promised cash flows is a promised IRR. It is what the WSJ prints.
The promised IRR is never used in the IRR Capital Budgeting Rule. For capital budgeting purposes, you must use an IRR computed from expected cash flows, not from promised cash flows.

Default (Credit) in Net Present Values

In PV applications, you have to use: E(Cash Flow) / [1 + E(Discount Rate)].

In the real world, users mess up PV calculations when they do not understand the distinction between expected and promised rates. You must use *expected values* in both numerator and denominator. The expected payoff is in the numerator, and it takes care of the default risk of our project. The PV of our loan promising $210 is:

$$PV = E(Cash\ Flow)/\,[1 + E(Discount\ Rate)] = \$208.40/(1 + 5\%) = \$198.47$$

It is not the promised amount of $210 divided by the cost of capital.

The expected discount rate – not the promised rate of return – is in the denominator. It is the opportunity cost of capital on other projects, quoted in terms of their expected rates of return, not in terms of their promised rates of return. You can't use the promised rate of return of opportunities elsewhere as your cost of capital.

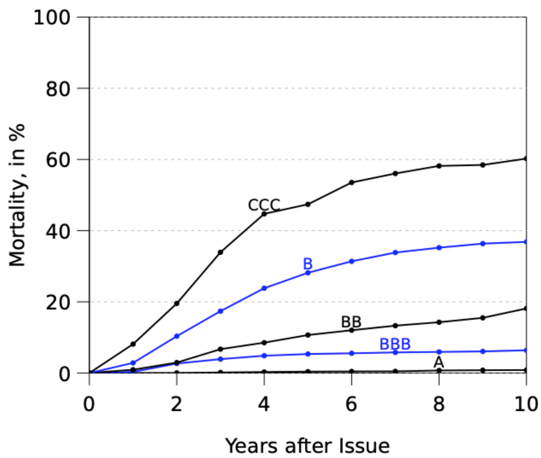
In a risk-neutral world, every project has the same denominating cost of capital (here 5%), regardless of how likely the project or bond is to pay what it promises.

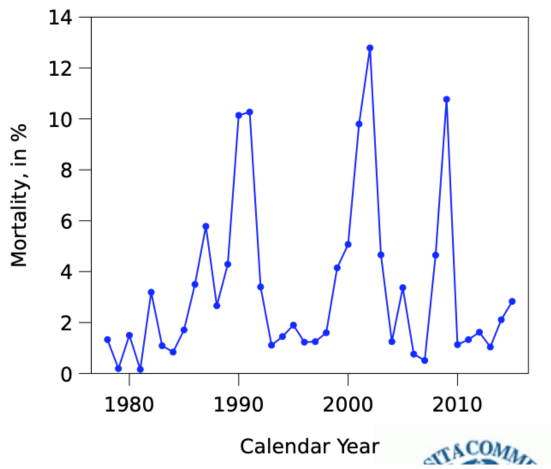
Credit Ratings

Large corporations have credit ratings, too, ranging from AAA (best) to F.

- Typical AAA firm has a close to 0% probability of default over 10 years.
- Typical B firm has a 20% probability of one non-payment over 5 years.
- Typical C firm has a 50% probability of one non-payment over 6-8 years.

U.S. History





More Default Risk or Risk Premium?

Most of the yield spread of corporate bonds is due to the chance of default (i.e., the credit spread). For example, if a Boston Celtics = 9.4%, whereas a similar Treasury = 5.6%, then I would guesstimate that the Celtics bond pays off, on average, say, about 6.0%. 3.4% is then the default risk, and 0.4% is the risk premium (including possibly the liquidity premium). Boston Celtics Credit Spread = 9.4% - 5.6% = 3.8%

The Single-Most Important Lesson Under Uncertainty

Never ever confuse **expected rates** with **(higher) promised rates**.

- The 9.4% from the Boston Celtics is not expected!
- Newspapers and websites virtually never report expected rates.

Credit (Default) Swaps (CDS)
Nowadays, you can buy insurance against default, called credit (default) swaps. The financial crisis of 2008 has made them famous. They played a central role.
This market is over-the-counter (OTC). Sellers are often hedge funds who want to speculate on default. Buyers are often mutual funds or pension funds who want to reduce their risk exposure. In the event of default, the seller of CDS may either have to pay the CDS buyer a fixed amount, or allow the CDS buyer to sell the bond for a pre-agreed price to the CDS seller upfront, as negotiated up front. However, default events – when the buyer of the CDS may demand payment – are declared by a committee of conflicted I-Bankers. Huh?

The Market for CDS
IN 2016, there was more than $17 trillion of single-name credit swaps outstanding. This is a rather opaque market – it is possible that the risk of credit is no longer with the holders of the corporate debt.

- Risk is somewhat similar to the housing derivative risk – an obscure bank in Germany may blow up over housing trouble in Kansas.
- A fund can buy the bonds, insure itself many times over against default with a CDS, and then vote to try to drive the firm itself into bankruptcy.

However, as a buyer of a CDS, you will also have to worry about whether the issuer of the CDS will go bankruptcy itself.

Uncertainty in CapBudg: PV with Debt and Equity
We have already covered rates of return and NPV under uncertain future cash flows. This book chapter represents another important conceptual leap:

- Introduces Payoff Tables and Contingent Claims Valuation
- Introduces Bonds vs. Levered Equity
- Introduces Bond Risk vs. Equity Risk

These are not minor topics, but some of the most important concepts in finance. They are big deals!

- Every investment opportunity in our perfect world is fairly priced.
- You can see yourself as one of two types of investors:
    A. A lender, who has provided capital in exchange for the promise of a fixed amount of money [called leverage], or
    B. A levered homeowner, (often just called homeowner), who owns the house with the bundled obligation to repay the loan.

A Financed Project (House/Firm/…)

Next Year's Payoffs	Probability	
$100	90%	(Sunshine)
$50	10%	(Hurricane)

The expected rate of return on 1-year Treasuries (and all other 1-year financial instruments) is 5%. The world is risk-neutral. This is the project example for all the following pages.

- What is the appropriate price for this project?
  Expected Payoff = $100 * 90% + $50 * 10% = $95
  Price = $P_0 = CF_1/(1+r)$ = Expected Payoff/(1 + 5%) = $90.48

- What is the rate of return on the project in the good state (= promised rate of return)?

Return = \$100/\$90.48 – 1 = + 10.53%

- What is the rate of return on the project in the bad state?
  Return = \$50/\$90.48 – 1 = - 44.74%

- What is the expected rate of return on the project? 5%
  Expected Rate of Return = 10.53% * 90% - 44.74% * 10% = 5%
  But we could also calculate it as: \$95/\$90.48 -1 = 5%

Levered Equity (Stock) + Risk-Free Bond
You can finance the project in one of two ways:
- You can buy it outright (with \$0 mortgage) with financing from your life's savings account.
- You can buy it with a mortgage and a smaller sum from your life's saving account. You then own just the residual equity called **Levered Equity** or **Levered Stock** or just **Stock**. It is what you get to keep after you will have repaid the debt.

Let's work with a specific example. Let's finance your purchase with a loan (= bond) promising \$50. We assume that financial markets are still perfect, as before.

- In the good state, how much do bond and levered equity receive?
  \$50 (Bond) & \$50 (Levered Equity), because in the good state, house is worth \$100, hance the bond is repaid fully (\$50) and the levered equity gets \$100 - \$50

- In the bad state, how much do bond and levered equity receive?
  The house is worth \$50, hence the bond is repaid fully (\$50) and the equity gets paid \$50 - \$50 = \$0.

The bond is essentially Risk-Free as it is fully repaid in both cases.

- What are the appropriate prices for the bond and the levered equity?
  Bond is Risk-Free so $P_{0(\text{Bond})}$ = \$50/ (1 + 5%) = \$47.62
  Levered Equity is not Risk-Free, $P_{0(\text{Equity})}$ = (\$50*90% + \$0*10%) / (1 + 5%) = \$42.86
  $P_{0(\text{Calculated in Previous Exercises})}$ = \$90.48
  $P_{0(\text{Bond})}$ = \$47.62, $P_{0(\text{Equity})}$ = \$42.86 and $P_{0(\text{Bond})} + P_{0(\text{Equity})}$ = \$90.48

This shows us that in a perfect financial market, Modigliani-Miller Theorem holds true.

The Main Diagram
- In the good state, what is the *rate of return* that the bond and the levered equity receive?
  $r_{0,1(\text{Bond|Good State})} = 5\%$
  $r_{0,1(\text{Equity|Good State})} = P_{1(\text{Equity|Good State})} / P_{0(\text{Equity})} - 1$ = \$50/\$42.86 = 16.67%

- In the bad state, what is the *rate of return* that the bond and the levered equity receive?
  $r_{0,1(\text{Bond|Bad State})} = 5\%$
  $r_{0,1(\text{Equity|Bad State})} = 0/P_{0(\text{Equity|Bad State})} - 1 = -100\%$

Hence, the Expected Equity return is: -100% * 10% + 16.67% * 90% = 5%
Or, alternatively, we can compute it as: $E(P_1)/P_0 - 1$ = (\$50 * 90% + \$0 * 10%)/\$42.86 – 1 = 5%

	Scheme 1	Scheme 2	
Project Payoffs	Firm, FM (=100% Equity)	Bond, DT (Promise=\$50)	Levered Equity, EQ
prob(G)=	r=	r=	r=
prob(B)=	r=	r=	r=
E(Payoff) (= E(C))			
E(Rate of Return) (= E(r))			
Discounted Price P_0			
% Financing			

- Is full project ownership (= zero leverage) or levered project ownership riskier?
  Full Ownership
    A. Expected Payoff: Good State → \$100 and Bad State → \$50
    B. Rate of Returns: Good State → 10.53% and Bad State → - 44.74%
  Levered Ownership
    C. Expected Payoff: Good State → \$50 and Bad State → \$0
    D. Rate of Returns: Good State → 16.6% and Bad State → - 100%

- Is full project ownership (= zero leverage) or bond ownership riskier? Full Ownership

Limited Liability
Limited Liability, you are on the hook only for what you invested, and no more. Limited Liability was a central innovation in finance, in the Renaissance. (It was not known in Medieval or Roman times). It came into wide use in the 18th and 19th century. It made it possible for owners to hand control to specialists and not worry for their entire holdings. The President of Columbia University wrote in 1911 that its discovery was more important than that of steam and electricity.

Bond Promising \$70 Next Year
Usually, equity has **limited liability**, which is how we will use it henceforth in the remainder of the course.
- Now price a bond with a promise of \$70.
- Enter everything you know.
- Work down the project without financing.
- Work down the pricing of the bond.
- Work back up the pricing of the equity.
- Use the next page.

(P.S.: this works exactly the same way with more than two possible outcomes)

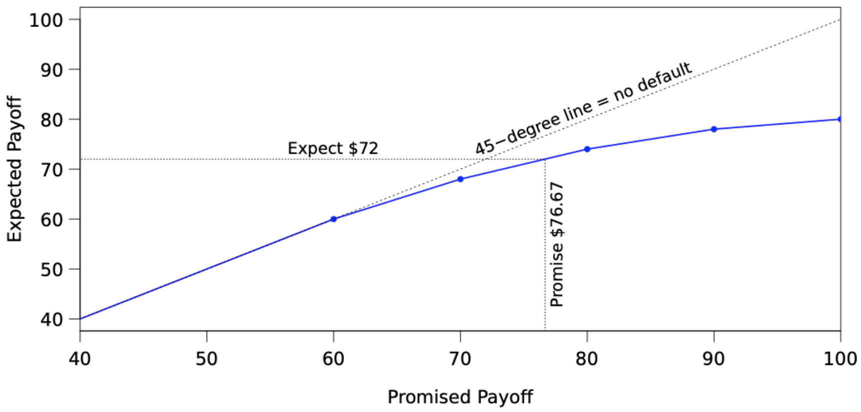
- What happens to the riskiness of the stock when more mortgage (say, \$70 rather than \$1) is taken on? The risk of the stock goes up, because the variability increases.

- What happens to the riskiness of the mortgage when more mortgage (say, \$70 rather than \$1) is taken on? The risk of the mortgage goes up.

- What happens to the riskiness of the "firm" (the house overall) when more mortgage is taken on? It stays the same due to Modigliani-Miller Theorem.

A Broader View of Leverage
Leverage = Small movement in lever can create much bigger or smaller movement elsewhere (in the equity).
Leverage = The safer part is "outsourced". Small movement in underlying project can make levered ownership much riskier – more upside and more downside.

More Than Two Possible Outcomes
Everything you learned generalizes



In fact, everything can be done with normally distributed returns, too. In this case, the curve would be smooth.

- Recall that you can discount nominal payouts with nominal expected rates of return and come to the same result as with real payouts with real expected rates of return.
- Can you discount promised payouts with promised rates of return and come to the same result as when you discount expected payouts with expected rates of return? No

Financial Data

Maintained Assumptions
In this part (consisting of three chapters), we maintain the assumptions of the previous chapter:
We assume perfect markets, so we assume four market features:

- No differences in opinion
- No taxes
- No transaction costs
- No big sellers/buyers – infinitely many clones that can buy or sell

We already allow for different rates of returns in each period.
We already allow uncertainty. So, we do not know in advance what the rates of return on every project are.
But we no longer assume risk-neutrality. We will henceforth allow for risk aversion.

First Part: A "Tour"
The goal of this part of the course is to summarize the basics of an investment course within the context of a broad finance course:

- Basic historical return patterns.
- What risk aversion does.
- How to measure risk and reward.
- The CAPM formula, its inputs, and how to use it.

The most important part of investments for our purposes is the determination of E[r] in the NPV formula. The book skips most derivations and, by doing so, misses important insights. **This is why I have written extra notes to supplement the book**. Of course, to become real financier, you really should take the Masters in Finance, and not just live with this summary.

Some Data
Asset Classes

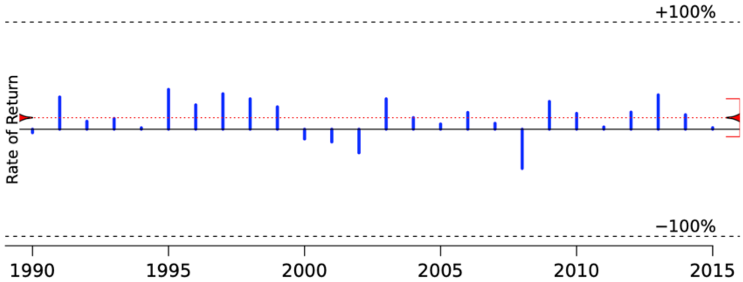
Asset classes are convenient portfolios that try to represent a swath of investment types – though inaccurately so. Examples are:

- **Stocks**: Large-firm stocks, Small-firm stocks, Foreign stocks, Value stocks, …
- **Bonds**: Long-term bonds, Risky bonds, Foreign bonds, Mortgage bonds, …
- **Short-Term**: Cash, Foreign Currency, Short-Term bonds, …
- **Real Estate**: Commercial, Retail, West-Coast, Russian, …
- **Art**: Paintings, Renaissance sculpture, Rare Books, …
- **Commodities**: Eggs, Bacon, Crude, …
- **Precious Metals**: Gold, Silver, Platinum, …
- **Agricultural**: Land, Grain, …

The S&P500 (with Dividends), 1970 – 2015

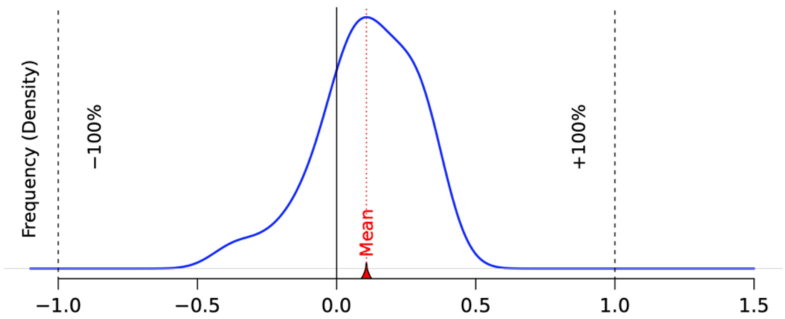
	Year				
Decade	0	1	2	3	4
2nd Part	5	6	7	8	9
1970	3.5%	14.1%	18.7%	−14.5%	−26.0%
1975	36.9%	23.6%	−7.2%	6.4%	18.2%
1980	31.5%	−4.8%	20.4%	22.3%	6.0%
1985	31.1%	18.5%	5.7%	16.3%	31.2%
1990	−3.1%	30.0%	7.4%	9.9%	1.3%
1995	37.1%	22.7%	33.1%	28.3%	20.9%
2000	−9.0%	−11.9%	−22.0%	28.4%	10.7%
2005	4.8%	15.6%	5.5%	−36.6%	25.9%
2010	14.8%	2.1%	16.0%	32.5%	13.5%
2015	1.5%	?			

Time – Series Plot, 1990 – 2015



Average Rate of Return = 10.7% per year
Annualized = 9.2% per year.

Histogram Plot, 1990 – 2015



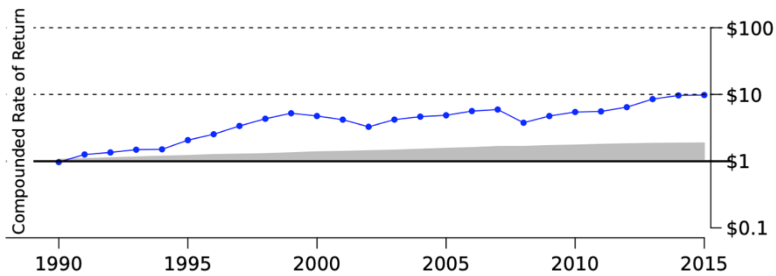
Average Rate of Return = 10.7% per year
Annualized = 9.2% per year
Standard Deviation = 12.6% per year.

- Is the average rate of return on an investment a good representation of the long-run rate of return that a buy-and-hold investor receives? No

- Compare two assets, A and B. They had equal average rates of return. However, A had a higher standard deviation than B. You are not risk-averse. Which investment would have earned you more money?

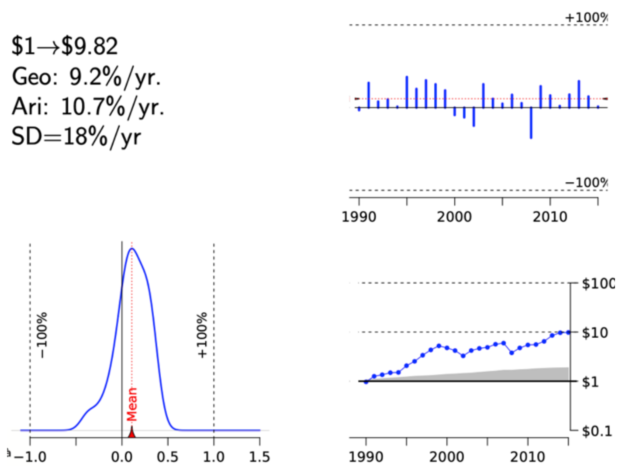
- Is it possible to lose all your money on a buy-and-hold portfolio that had a positive average rate of return?

Cumulative Performance Plot for Stocks, 1990 – 2015



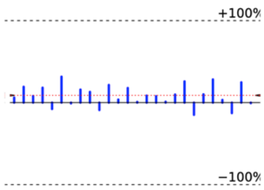
End Result: $1 in Jan. 1990 became approx. $10 in Dec. 2015.

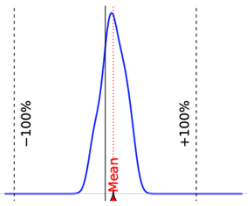
Stocks

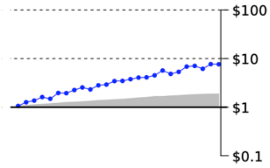


Bonds

$1→$7.64
Geo: %/yr.
Ari: 8.1%/yr.
SD=13%/yr

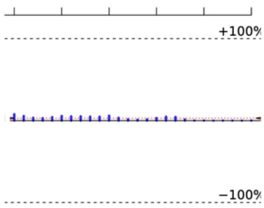


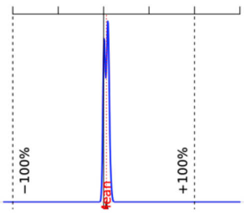


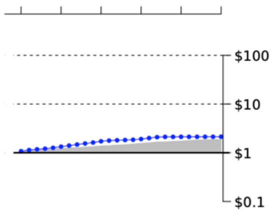


Cash

$1→$2.14
Geo: %/yr.
Ari: 3.0%/yr.
SD=2.4%/yr

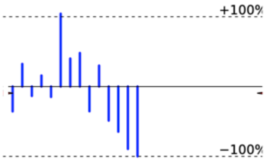


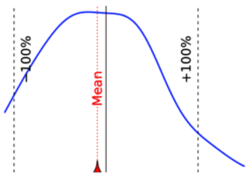


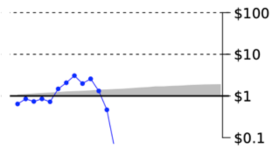


Stock UAL

$1→$0
Geo: %/yr.
Ari: –9.4%/yr.
SD=8%/yr

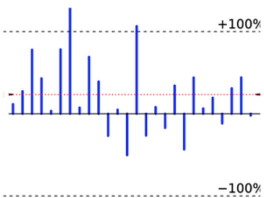


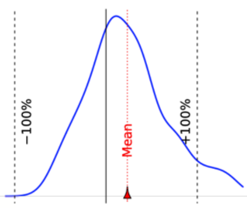


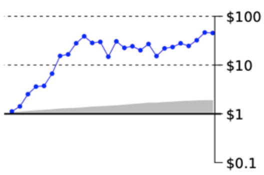


Stock INTC

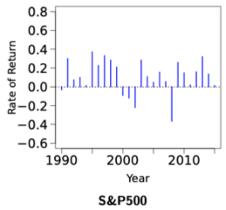
$1→$45
Geo: %/yr.
Ari: 23%/yr.
SD=18%/yr

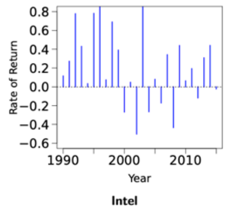


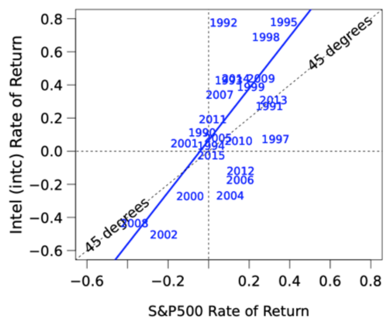




Correlation and (Market-) Beta







- Which asset classes generally offer higher average rates of returns? Stocks
- Which asset classes (and stocks) were riskier? Stocks (Small)
- Could you have lost your shirt? Yes
- Is there a risk-return relationship? Yes
- Do assets with a positive average rate of return always make you money? No
- Do stocks move together? Intuitively, can we exploit any non-synchronicity? Yes & Yes
- Is there anything special to multiple-stock investments? Yes
- Can you Trust History? Yes some

History vs. Future – Again
Finance has one huge advantage relative to other fields of economics – we have lots of data! Statisticians often pretend historical distribution (means, standard deviations, betas, etc.) are representative of the future distribution. That is, one should not pretend that we can judge a powerball gamble's outcome by how it did last week, but that we can judge a powerball risk and reward by how it did over the last many thousands of weeks. If we know the physics of ball drawing, we don't even need any history. We can then work out the expected risk and expected reward mathematically. Alas, we do not know the underlying physics of financial investments, so we work with historical data. Historical data is helpful, but it can also mislead if it is not used carefully. Correlations and variances are more "stable" ("reliable") than historical average rates of returns. Tail risk is tough. The only reason why we use historical data is because the alternative is no data and this would be worse.

Geometric and Arithmetic Returns: How To Extrapolate Over 20 Years?
How do you "roughly" translate geo to ari returns and vice-versa? If rates of returns are approximately normally distributed, then the ari mean is higher than the geo mean by about half the variance. Stocks here: 10.7% - 18%ˆ2/2 = 9.1%. Correct = 9.2%.

There are some not-so-obscure issues how to think about historical rates of return. Explained in the book. Often neglected. Tricky statistical problem.

Causality vs. Correlation

This is one of the most important questions in finance, economics, and business. Does correlation mean causation? 80% of consultant reports get this wrong – often deliberately to fool their clients. Regression Discontinuity is unusually good at empirical evidence, but not all questions can be addressed by it.

Market Institutions

- Brokers: Retail B vs. Prime B. (Execution and Margin).
- Market vs. Limit orders.
- Various modifications: Fill-or-kill, Good for the day, etc.
- Exchanges and non-Exchanges. In-person or computerized, batched auction or continuous, electronic crossing. OTC. (Pink sheets).
- Regulation: Congress, SEC, Exchanges(?!).
- Seeing the order book is huge advantage.
- Mutual Funds (more funds than stocks today!).
- Open-end vs closed-end distinction.
- Investment companies under the 1940 Act: UITs, open-end=mutual fund in the US, closed-end.
- Many other investment vehicles, e.g., hedge funds, private equity funds, venture capital funds, ADRs, trust funds, etc.
- Entry of corporate securities into the financial markets: IPOs, underwriters, reverse mergers, SEOs.
- Exit of corporate funds from the financial markets: Dividends, repurchases, delisting, limited liability, financial distress.

Diversification

A Digression – The Egg Approach to Investing

"*Don't put all your eggs in one basket ... **Diversify***".

Egg Merchandise

The insights of investments apply to business products, just as they apply to financial investments.
Your problem: Choose a portfolio of products (eggs) to bring to the market for sale:

- You do not know before you stock your basket which eggs will sell – but you have a good idea which eggs tend to sell better than others on average.
- Some products have higher likelihood of selling, others have lower likelihood of selling.
- For your customers, egg products have a fashion aspect – some types will be highly desirable, others less so.
- They could be imperfect substitutes for one another.

We assume perfect markets: there are a large number of sellers and buyers.
Let's presume you care about your overall risk and reward.

- Do you care about overall basket risk and reward, or do you care about the risk and reward of each egg in your basket individually? Overall

- Should you purchase just the most likely egg to sell?

- Should you go out of the business entirely?

- Should you purchase a mix of different products? Yes

- Consider a completely different type of product, which is very risky in itself. That is, you do not think it will sell. If you have purchased just the most likely product seller, what is your risk of having one completely different product in your basket?

- (Equilibrium) How much would you be willing to pay for the completely different type of product on the margin?

Eggs and Markets
First, you have to understand portfolio selection. Given the prices and probabilities of when what eggs will sell, how would you optimally stock your basket? If an asset is very different from all other assets (it pays off when other assets do not pay off), then you are willing to buy this asset for a higher price (expecting a lower return) than you would otherwise.

In the context of investment theory, you should like assets that (a) pay off often; and (b) pay off big when other assets do not. This topic is called **optimal portfolio choice**.

Then, after you have understood what everyone wants, you have to understand how the prices of eggs will adjust to the many people that want to stock baskets. If every seller wants the rare purple egg because it is so different from all the other eggs, then its price will be bid up.

The CAPM is the model that tells us how risk and reward are related to:
- How individual products are expected to perform.
- How individual products are different from others.

If you are a corporate CEO, you must think about what eggs you want to make, i.e., what projects you want to offer to investors that like eggs that:
- Pay off often
- Pay off when other eggs do not.

Takeaways
- Different asset classes offer different combinations of risk and expected returns.
- Diversification helps reduce risk (up to some point, to be made precise in later chapters) while keeping expected return fixed.

Preferences
Maintained Assumptions
In this part (consisting of three chapters), we maintain the assumptions of the previous chapter:
We assume perfect markets, so we assume four market features:
- No differences in opinion
- No taxes
- No transaction costs
- No big sellers/buyers – infinitely many clones that can buy or sell

We already allow for different rates of returns in each period.
But now, we allow uncertainty. So, we do not know in advance what the rates of return on every project are. We need to predict (describe) the future. For this, we need statistics.

Preferences in General
In economics we define preferences by a **utility function**, which is a mapping from a bundle of consumption goods (i.e., a vector) **c** = (c_1, …, c_k) to the real line, U(**c**) ∈ ℝ where a bundle $c_1 \succ c_2$ if and only if $U(c_1) > U(c_2)$.

Optional actions are chosen to maximize utility. In general, consumption goods can be anything.

- Examples, food, cloth, car, housing, leisure time, philanthropy, family, relationships, etc.
- If we are interested in consumption at different points in time and in uncertain states of the world, we can define for each state and point in time a separate consumption good.

In finance, we assume that we can assign a price $\mathbf{q} = (q_1, \ldots, q_k)$ to each consumption good.
Examples: price of wine, rent, foregone salary due to a relaxing day at home, donation to a charity, education of a child, etc.
Suppose you have initial wealth W, then any feasible consumption bundle **c** satisfies $\boldsymbol{q}^T\boldsymbol{c} \leq W$.
We further assume that U(**c**) is increase in good c_i if and only if $q_i > 0$.
Example ($q_1 > 0$): I enjoy a relaxing day at home (define as c_1 with U(**c**) increasing in c_1) but then I do not earn any salary which is costly ($q_1 > 0$).
Example ($q_1 < 0$): I do some extra work which is exhausting (define as c_2 with U(**c**) decreasing in c_2) but I earn some extra money ($q_2 < 0$).

Suppose bundle **c*** is the optimal consumption choice given wealth W, we define $u(W) = U(\mathbf{c}^*) = \max_{\mathbf{c} \text{ s.t. } \mathbf{q}^T\mathbf{c} \leq W} \{U(\mathbf{c}^*)\}$ as the maximum level of utility we can achieve with wealth W.
Suppose I receive some helicopter money $\Delta W > 0$. I still can choose **c*** if I prefer so ($\boldsymbol{q}^T\boldsymbol{c} \leq W < W + \Delta W$, which was not feasible before.
Thus, $u(W) \leq u(W + \Delta W)$, that is, u(W) is at least weakly increasing in W (we often assume it is strictly increasing).

Working directly with u(W) simplifies our problem substantially! I.e., we will choose an optimal wealth allocation to consumption and risky assets to maximize u(W) and we do not worry about the particular allocation to diverse consumption goods.
It is a reasonable simplification for finance where we only care about the allocation of wealth/ optimal consumption and portfolio choice; but, it may not be useful for other applications in economics.
If we have uncertainty in wealth, that is, $W(s) = W_0 + D(s) = W_0R(s)$ is state-dependent, due to risky returns R(s) or uncertain future income D(s), then popular assumptions to maximize expected utility E[u(W)] = $\sum_s p_s\, u(W(s))$.

What is the functional form of u()?

- U(W) is increasing in W, that is, $\frac{\partial u(W)}{\partial W} > 0$. People prefer more wealth to less.
- What else can we learn about the shape of u() from people's actions?

St. Petersburg Game
Players pay an entry fee into the game, which will only be played once.
The game consists of tossing a fair coin until the first head appears.
The number of tails (n) that appears until the first head is tossed is used to compute the payoff to the participant as $R(n) = 2^n$.
The entry fee is the only money the player can lose.

How much would you pay this game?

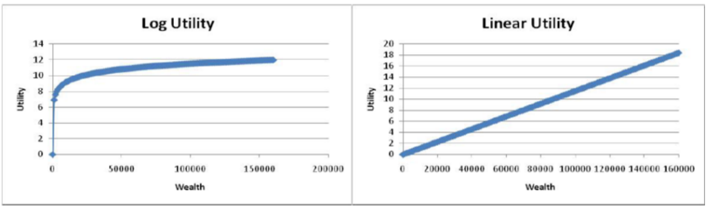
Daniel Bernoulli studied risk and return of this problem in the early 1700's.

Tosses	Outcome	Tails	Prob.	\$ Payoff	Prob. × Payoff
1	H	0	0.500	\$1.00	0.50
2	T,H	1	0.250	\$2.00	0.50
3	T,T,H	2	0.125	\$4.00	0.50
4	T,T,T,H	3	0.063	\$8.00	0.50
N	T,....,T,H	N−1	$\left(\frac{1}{2}\right)^N$	$\$2^{N-1}$	0.50

Expected payoff is $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{1+n} 2^n = \sum_{n=0}^{\infty} 0.5 = \infty$

If you only care about expectations, you should pay everything you own and more to purchase the right to take this gamble! Yet, in practice, no one is prepared to pay such a high price. Why? Even though the expected payoff is infinite, the distribution of payoffs is not very attractive.
- With 93% probability we get $8 or less.
- With 99% probability we get $64 or less.

Bernoulli realized that (internally) people didn't assign the same value to all the payoffs, and suggested that large gains should be weighted less.
- Bernoulli suggested natural logarithm is ln(2).
- Bernoulli suggested one should pay at most $2.



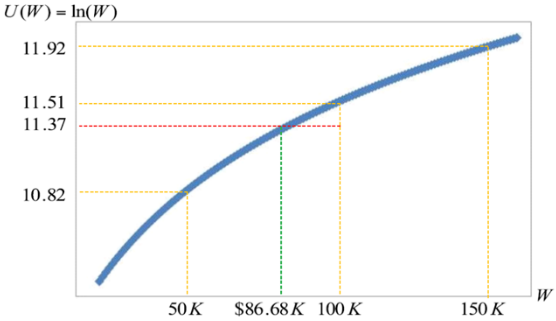
Risk Aversion
Bernoulli's observation is what we call risk aversion. Risk aversion means that u(W) is concave, that is, $\frac{\partial^2 u(W)}{\partial W^2} < 0$, by Jensen's inequality $E[u(W)] \leq u(E[W])$.
Formally, Arrow-Pratt measure of risk aversion is defined as:

$$RA = -\frac{\frac{\partial^2 u(W)}{\partial W^2}}{\frac{\partial\, u(W)}{\partial W}}$$

Large RA means that bad outcomes (low W(s)) get a larger weight than good outcomes (large W(s)) in the expected utility calculation, so an investor is willing to pay little for a gamble or equivalently needs a large risk premium as compensation.

Example of log-utility: RA = $-\frac{-\frac{1}{W^2}}{\frac{1}{W}} = \frac{1}{W}$; if we are more wealthy, we are less risk averse and more willing to pay for a gamble.

Computation with Log-Utility



Example: Expected utility is $\boldsymbol{E[u(W)] = E[u(W_0 + D_i)] = \sum_{s\in\{G,B\}} p_s ln(W_0 + D_i(s))}$
Would you rather accept a performance based salary of $150k if performance is high (prob. 0.5) and $50k if performance is low; or a fixed salary of $100k?
- Expected utility of performance based salary: E[u(W)] = 0.5*ln(0+150,000) + 0.5*ln(0+50,000) = 11.37
- Expected utility fixed salary: E[u(W)] = ln(0+100,000) = 11.51

Hence, you would choose fixed salary.

What is the lowest fixed salary you are willing to accept over the performance based salary?

- Indifferent between performance based salary and fixed salary of \$X such that: $\ln(0 + X) = 0.5 * \ln(0 + 150{,}000) + 0.5 * \ln(0 + 50{,}000) = 11.37$ or X = \$86.68k
- We call this the **certainty equivalent** of a risky gamble (what sure payoff is equivalent to the uncertain payoff).

Formally, suppose initial wealth $\boldsymbol{W_0}$, certainty equivalent CE of payoff $\boldsymbol{D_i}$ solves

$$\boldsymbol{u(W_0 + CE) \;=\; E[u(W_0 + D_i)]}$$

Final remark:
- Analysis depends crucially on the level of the initial wealth, which is equivalent to the observation that RA = 1/W depends on the wealth level.
- However, if we compare two returns R_1, R_2 on our entire wealth, that is, payoffs R_1W_0 vs R_2W_0 (instead of fixed payoffs D_F vs D_P, which are independent of our initial wealth), then $R_1W_0 \succ R_2W_0$ if and only if $R_1(W_0 + \Delta W) \succ R_2(W_0 + \Delta W)$.
  This is because log-utility allows to separate initial wealth from the return, $u(R_1 W_0) = \ln(R_1 W_0) = \ln(R_1) + \ln(W_0)$ and thus $\mathbb{E}[u(R_1 W_0)] > \mathbb{E}[u(R_2 W_0)] \Leftrightarrow \mathbb{E}[\ln(R_1)] > \mathbb{E}[\ln(R_2)]$ $\Leftrightarrow \mathbb{E}[u(R_1(W_0 + \Delta W))] > \mathbb{E}[u(R_2(W_0 + \Delta W))]$ is independent of the initial wealth level W_0 or $W_0 + \Delta\ W$

What is missing from this representation? Incentives!

The Mean-Variance Case

Revisiting the Functional Form of u()

Functional form of u():
- Prefer more wealth to less: $\frac{\partial\, u(W)}{\partial W} > 0$
- Risk Aversion: $\frac{\partial^2 u(W)}{\partial W^2} < 0$

What else can we say?
- In the previous example (computation with log-utility) we saw that initial wealth is important to compute expected utility.
- Remember log-utility implies that risk aversion RA = 1/W is decreasing in wealth… Is that a desirable property?

Risk Aversion and Initial Wealth

You have to choose between:
- 0% risk-free rate of return
- Investment with 80% probability to return +100%, and with 20% probability to return -100%.

Suppose you are investing \$10 (payoffs: \$20 or \$0). Which would you choose between A and B?
Suppose you are investing \$100,000 (payoffs: \$200,000 or \$0). Which would you choose between A and B?
Suppose you are investing your entire current wealth and all your income over the next 30 years (Double your wealth and future income or nothing). Which would you choose between A and B?

Decreasing Risk Aversion

Most people are more willing to risk small amounts relative to their wealth than relatively large amounts. RA is decreasing in W!
We define relative risk aversion as RRA = W * RA
- Constant RRA is popular in finance for preferences with decreasing RA (= RRA/W) in wealth.

- Under constant RRA our preference over a return on our entire initial wealth is independent of the level of our initial wealth.
- Some popular preferences in finance are constant RRA: $u(W) = \begin{cases} \frac{W^{1-RRA}}{1-RRA} & \text{for } RRA \neq 1 \\ \ln(W) & \text{for } RRA = 1 \end{cases}$

Mean-Variance Preference as an Approximation
Instead of defining utility over consumption or wealth, **Mean-Variance Preferences** are defined over moments of returns,

$$u_{MV}(r_i) = \mathbb{E}[r_i] - \frac{\gamma}{2} Var[r_i] = \mu_i - \frac{\gamma}{2}\sigma_i^2$$ Where γ captures risk aversion.

Justification (A): If returns R_i are normally distributed then R_iW is also normally distributed, and we can rewrite any (constant RRA) utility function of R_iW as a function of the expectation (or mean) and variance of r_i (remember: $R_i = 1 + r_i$).
- Remember: a normal distribution is fully characterized by its expectation and variance.

Justification (B): If returns $r_i = R_i - 1$ are small, then we have the following approximation:

$$\begin{aligned} \mathbb{E}[u(R_iW)] &\approx \mathbb{E}\left[u(W) + \frac{\partial u(W)}{\partial W}(R_iW - W)\right] \\ &\quad + \mathbb{E}\left[\frac{1}{2}\frac{\partial^2 u(W)}{\partial W^2}(R_iW - W)^2\right] \\ &= u(W) + \frac{\partial u(W)}{\partial W}\mathbb{E}[r_i]\,W + \frac{1}{2}\frac{\partial^2 u(W)}{\partial W^2}\mathbb{E}\left[r_i^2\right] W^2 \end{aligned}$$

and thus, $R_1W \succ R_2W$ iff $\mathbb{E}[u(R_1W)] > \mathbb{E}[u(R_2W)]$ iff

$$\mathbb{E}[r_1] + \frac{1}{2}W\frac{\frac{\partial^2 u(W)}{\partial W^2}}{\frac{\partial u(W)}{\partial W}}\mathbb{E}\left[r_1^2\right] > \mathbb{E}[r_2] + \frac{1}{2}W\frac{\frac{\partial^2 u(W)}{\partial W^2}}{\frac{\partial u(W)}{\partial W}}\mathbb{E}\left[r_2^2\right]$$

(continued) or approximately

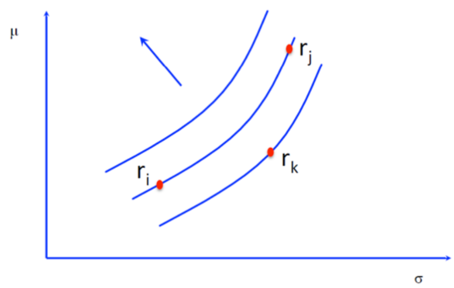
$$\mu_1 - \frac{RRA}{2}\sigma_1 > \mu_2 - \frac{RRA}{2}\sigma_2$$

or

$$u_{MV}(r_1) > u_{MV}(r_2)$$

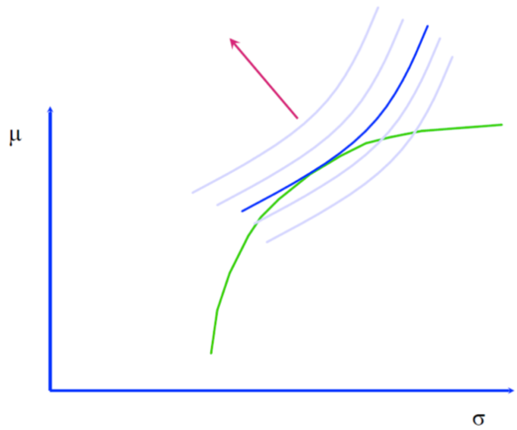
with $\gamma = RRA = W\frac{\frac{\partial^2 u(W)}{\partial W^2}}{\frac{\partial u(W)}{\partial W}}$ and $\sigma_i^2 = \mathbb{E}\left[r_i^2\right] - \mathbb{E}[r_i]^2 \approx \mathbb{E}\left[r_i^2\right]$

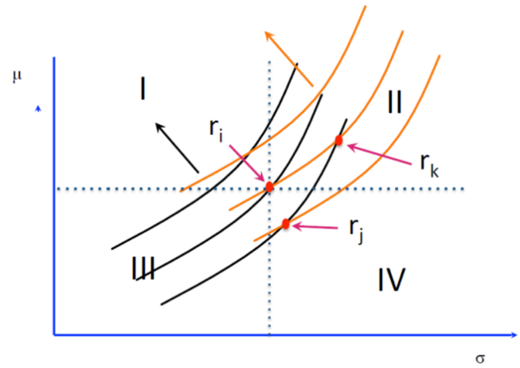
Mean-Variance Preferences – Indifference Curves



$r_i \sim r_j \succ r_k$

Indifference Curves of Another Investor & Mean-Variance Dominance





r_i mean-variance dominates r_j

(Black: $r_i \succ r_k \sim r_j$; Orange: $r_i \sim r_k \succ r_j$)

Behavioral Preferences

Failure of Mean-Variance Preferences

Suppose you have mean-variance preferences with risk aversion coefficient $\gamma = 10$.
There are two state of the world: G and B with $p_G = Pr\{G\} = 0.01, p_B = Pr\{B\} = 0.99$.

Do you prefer:

- Asset 1 with risk-free rate of return $r_1(G) = r_1(B) = 0$.
- Asset 2 with return $r_2(g) = 10\%$ or $r_2(B) = 0$.
- Asset 3 with return $r_3(G) = 25\%$ or $r_3(B) = 0$.

Answer:

$$
\begin{aligned}
u_{MV}(r_1) &= 0 - \frac{10}{2}0^2 = 0 \\
u_{MV}(r_2) &= 0.001 - \frac{10}{2}\left(0.01\,(0.1 - 0.001)^2 + 0.99\,(0 - 0.001)^2\right) \\
&= 0.00051 \\
u_{MV}(r_3) &= 0.0025 - \frac{10}{2}\left(0.01\,(0.1 - 0.0025)^2 + 0.99\,(0 - 0.0025)^2\right) \\
&= -0.00059 \\
r_2 &\succ r_1 \succ r_3
\end{aligned}
$$

But note: r_3 always (weakly) outperforms r_2, and r_2 always (weakly) outperforms r_1 → first order stochastic dominance. Mean-Variance preferences may dislike a free-lunch? The reason is two-fold: normality is violated and returns are large.

Digression – Rationality in Economics

Key Axioms for Von Neumann-Morgenstern utility (expected utility):

(1) **Completeness**: for any R_i, R_j, either $R_iW \succeq R_jW$ or $R_jW \succeq R_iW$

(2) **Transitivity**: for returns R_i, R_j, R_h, if $R_iW \succeq R_jW$ and $R_jW \succeq R_hW$, then $R_iW \succeq R_h\,W$

(3) **Continuity**: for returns R_i, R_j, R_h with $R_i\,W \succeq R_jW \succeq R_h\,W$, there exists a constant $\theta \in (0,1)$ such that $\theta R_iW + (1-\theta)R_hW \succeq R_jW$

(4) **Independence**: for returns R_i, R_j, R_h and some constant $\theta \in (0,1)$, if $R_iW \succeq R_jW$, then $\theta R_iW + (1-\theta)R_hW \succeq \theta R_jW + (1-\theta)R_h\,W$

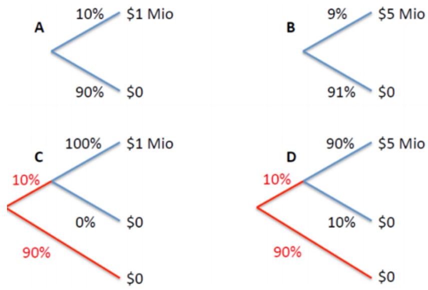
Digression – Allais Experiment

What do you prefer?

A. \$1 Mio with probability 10%, \$0 otherwise
B. \$5 Mio with probability 9%, \$0 otherwise

What do you prefer?

C. \$1 Mio for certain
D. \$5 Mio with probability 90%



Allais Paradox and Failure of Rationality
Note, Independence axiom implies that **A** $\succ$ **B** if and only if **C** $\succ$ **D**.
If you choose **B** $\succ$ **A** and **C** $\succ$ **D**, then you are not "rational" in the expected utility sense!

Digression – Ellsberg's Paradox
There is an urn with 30 balls and I randomly pick one

- 10 red balls
- 20 other balls
    A. Some of the 20 balls are black some are yellow.
    B. You do not know the ratio between black and yellow balls.

- What do you prefer?
    A. You get \$100 if I draw a red ball.
    B. You get \$100 if I draw a black ball.

- What do you prefer?
    C. You get \$100 if I draw a black or yellow ball.
    D. You get \$100 if I draw a red or yellow ball.

Independence axiom implies that **A** $\succ$ **B** if and only if **D** $\succ$ **C**.

- **A** $\succ$ **B** implies:

$$\begin{aligned} & p_R u(W+100) + [1-p_R]\, u(W) \\ > \quad & p_B u(W+100) + [1-p_B]\, u(W) \\ \iff \quad & p_R [u(W+100) - (W)] > p_B [u(W+100) - (W)] \end{aligned}$$

- **C** $\succ$ **D** implies:

$$\begin{aligned} & [p_B + p_Y]\, u(W+100) + [1-p_B-p_Y]\, u(W) \\ > \quad & [p_R + p_Y]\, u(W+100) + [1-p_R-p_Y]\, u(W) \\ \iff \quad & p_B [u(W+100) - (W)] > p_R [u(W+100) - (W)] \end{aligned}$$

Which represents a contradiction.
If you choose **A** $\succ$ **B** and **C** $\succ$ **D**, then you are not "rational"!

Beyond Expected Utility Theory
Motivated by both Allais's and Ellsberg's Paradoxes, as well as a series of experiments of their own, Kahneman and Tversky departed from Expected Utility Theory and built **Prospect Theory**, the modern theory of decisions under risk.
The assumptions and predictions of Prospect Theory have been tested numerous times and found consistent with people's behavior in many laboratory experiments.

Prospect Theory
Prospect Theory is based on four pillars:

- **Reference Dependence**, people discard components that are shared by all prospects under consideration → People derive utility from gains or losses **relative to some reference point**.
- **Loss Aversion**, people are more sensitive to losses than to gains of similar magnitude.
- **Diminishing Sensitivity**, going from a gain (loss) of \$100 to a gain (loss) of \$200 impacts utility more than going from a gain (loss) of \$1,100 to a gain (loss) of \$1,200 → People are risk averse in the domain of gains, but risk seekers in the domain of losses.

- **Probability Weighting**, people do not weight outcomes by their objective probabilities p_i but rather by transformed probabilities or decision weights, π_i.

Prospect Theory More Formally

If a gamble promises outcome x_i with probability p_i, Tversky and Kahneman (1992) propose that people assign the gamble the value.

$$\sum_i \pi_i v(x_i)$$

where

$$v = \begin{cases} x^{\alpha} & \text{if} \quad x \geq 0 \\ -\lambda(-x)^{\alpha} & \text{if} \quad x < 0 \end{cases}$$

and

$$\pi_i = w(P_i) - w(P_i^*)$$

$$w(P) = \frac{P^{\gamma}}{(P^{\gamma} + (1-P)^{\gamma})^{1/\gamma}}$$

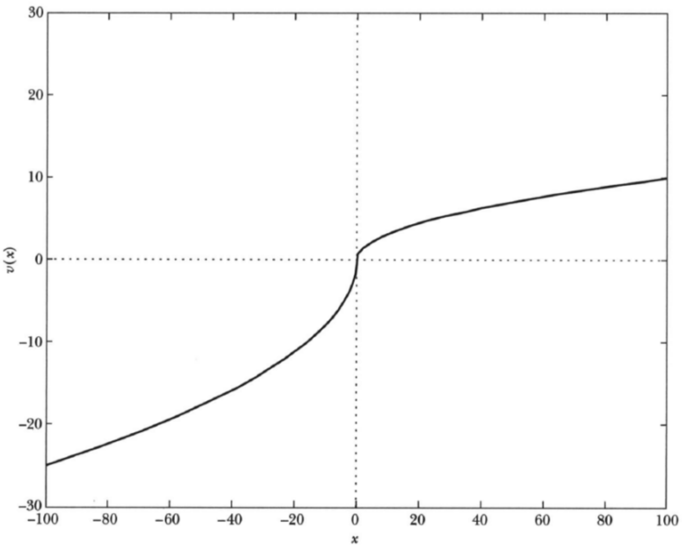
where P_i (P_i^*) is the probability that the gamble will yield an outcome at least as good as x_i

Tversky and Kahneman (1992) use experimental evidence to estimate $\alpha = 0.88, \lambda = 2.25$ and $\gamma = 0.65$.

Note that λ is the coefficient of loss aversion often estimated to be in the neighborhood of 2.

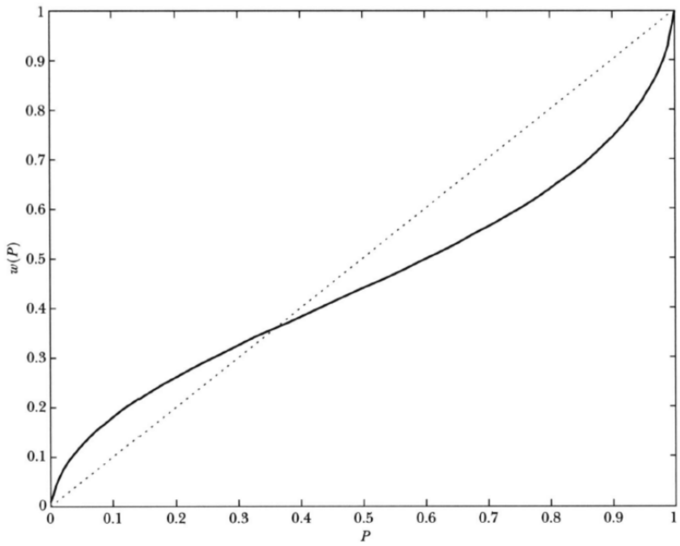
Prospect Theory – The Value Function & Probability Weighting

Figure 1
The Prospect Theory Value Function



Notes: The graph plots the value function proposed by Tversky and Kahneman (1992) as part of cumulative prospect theory, namely $v(x) = x^{\alpha}$ for $x \geq 0$ and $v(x) = -\lambda(-x)^{\alpha}$ for $x < 0$, where x is a dollar gain or loss. The authors estimate $\alpha = 0.88$ and $\lambda = 2.25$ from experimental data. The plot uses $\alpha = 0.5$ and $\lambda = 2.5$ so as to make loss aversion and diminishing sensitivity easier to see.

Figure 2
The Probability Weighting Function



Notes: The graph plots the probability weighting function proposed by Tversky and Kahneman (1992) as part of cumulative prospect theory, namely $w(P) = P^{\delta}/(P^{\delta} + (1-P)^{\delta})^{1/\delta}$, where P is an objective probability, for two values of δ. The solid line corresponds to $\delta = 0.65$, the value estimated by the authors from experimental data. The dotted line corresponds to $\delta = 1$, in other words, to linear probability weighting.

Prospect Theory – Assessment

Prospect Theory has been a building block of Behavioral Finance; we will discuss Behavior Finance in more detail in the class on Market Efficiency. For now, we shall continue to operate under Expected Utility Theory and derive its implications for Finance, including CAPM and APT among others. These are currently active areas of research.

Takeaways

- We use statistics notation and definition to deal with uncertainty.
- Matrix algebra helps deal with portfolio allocation problems.
- In finance investors care about maximizing wealth.