BEMACS Bocconi

30401 MATHEMATICS AND STATISTICS - MODULE 2 (STATISTICS)

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EXERCISES ON COMBINATORICS AND ELEMENTARY PROBABILITY

(1) a) How many different 7-place licence plates are possible if the first 2 places are for letters and the other 5 for numbers?
b) repeat part (a) under the assumption that no letter or number can be repeated in a single plate.

(Answer: $26^2 \cdot 10^5$ a) ; b) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$)

(2) How many outcome sequences are possible when a die is rolled four times?

(Answer: 6^4)

(3) Twenty workers are to be assigned to twenty different jobs, one to each job. How many different assignments are possible?

(Answer: 20!)

(4) A child has 12 blocks, of which 6 are black, 4 are red, 1 is white and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

(Answer: $\frac{12!}{6!4!}$)

(5) John, Jim, Jay and Jack have formed a band consisting of 4 instruments.
a) If each of the boys can play all 4 instruments, how many different arrangements are possible?
b) what if John and Jim can play all four, but Jay and Jack can each play only drums and piano?

(Answer: a) 4!; b) 4)

(6) 4 different books are to be distributed among 7 children.
a) How many possible outcomes?
b) How many, if no child can receive more than one book?

(Answer: a) 7^4; b) $\frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4$)

(7) Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many outcomes are possible if
a) a student can receive any number of awards?
b) each student can receive at most 1 award?

(Answer: a) 30^5; b) $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 = 17100720$)

(8) Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

(Answer: $\binom{20}{2} = 190$)

(9) A dance class consists of 22 students, 10 women and 12 men. If 5 men and 5 women are to be selected and then paired off, how many choices are possible?

(Answer: $\binom{10}{5}\binom{12}{5}5! = 23950080$)

(10) 7 different gifts are to be distributed to 10 children. How many distinct results are possible if no child is to receive more than one gift?

(Answer: $\frac{10!}{3!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$)

(11) A committee of 7 people, consisting of 2 Republicans, 2 Democrats and 3 Independents is to be chosen from a group of 5 Republicans, 6 Democrats and 4 Independents. How many committees are possible?

(Answer: $\binom{5}{2}\binom{6}{2}\binom{4}{3} = 600$)

(12) A person has 8 friends, of whom 5 will be invited to a party.
   a) How many choices are there if 2 of the friends are feuding and will not attend together?
   b) How many choices are there if 2 of the friends will only attend together?

(Answer: a) 36; b) 26)

(13) Six friends are waiting to be seated in a restaurant.
   a) How many possible arrangements if they will be seated at a bar?
   b) How many possible arrangements if they will sit at a round table?
   c) Which is the probability that Carlo sits next to Francesca in the two cases?

(Answer: a) $\frac{6!}{2}$; b) $\frac{5!}{2}$; c)$\frac{1}{3}$ and $\frac{2}{5}$)

(14) A certain town has 3 newspapers: A, B and C. The proportions of townspeople who read these papers are as follows: A 20%, B 16% and C 14%. Moreover, 8% of townspeople reads both A and B, 5% reads both A and C, 4% reads both B and C and, finally, 2% reads the 3 newspapers. Compute the probability that
   a) a person randomly chosen in town reads at least a newspaper;
   b) a person randomly chosen in town does not read any;
   c) a person randomly chosen in town reads exactly one newspaper.

(Answer: 35%; 65%; 22%)

(15) (Birthdays' problem) If k people are present in a room (with $2 \leq k \leq 365$), what is the probability that at least 2 of them celebrate the birthday on the same day of the year? (suppose no bissextile)?

(Answer: $1 - \frac{365!}{365^k(365-k)!}$)

(16) An urn contains n balls, one of which is special. If k of these balls are withdrawn one at a time without replacement, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

(Answer: $\frac{k}{n}$)

(17) An urn contains 6 white and 5 black balls. 3 are chosen at random all at once. What is the probability one is white and the other 2 black?

(Answer: $\frac{4}{11}$)

(18) Braille characters are small rectangular blocks called cells that contain tiny palpable bumps called raised dots. The number and arrangement of these dots distinguish one character from another. Any character is formed by transforming in a raising dot at least one of the six points of a grid formed by 3 horizontal and 2 vertical parallel lines.
   a) How many Braille characters can be formed?
   b) How many with 4 raised dots?
   c) Chosen a 4 raised dots character, what is the probability that the 4 dots form a rectangle or a square?

(Answer: a) 63; b) 15; c) $\frac{1}{5}$)

(19) We have 5 pairs of shoes. We select 4 shoes at random, which is the probability that
   a) there is no complete pair of shoes;
   b) there is at least a complete pair of shoes;
   c) there is exactly one complete pair of shoes.

(Answer: a) $\frac{8}{21}$; b) $\frac{13}{21}$; c) $\frac{12}{21}$)

(20) Consider a grid of points formed by 4 horizontal and 5 vertical parallel lines. Suppose that, starting from the point A (left-bottom corner), you can go one step up or one step right at each move. This procedure is continued until the point labeled B (right-up corner) is reached.
   a) How many different paths are possible?
   b) If any of them is equally likely, what is the probability of passing through the point P $(2, 2)$.

(Answer: a) 35; b) $\frac{18}{35}$)

(21) A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first.
(Answer: $\frac{2}{5}$)

(22) We would like to count how many 8-character passwords can be created using the following rules: the 8 characters have to be different and contain at least 2 numbers and 3 letters.

(Answer: $8! \sum_{k=2}^{5} \binom{10}{k} \binom{21}{8-k}$)

(23) 60% of the students at a certain school wear neither a ring nor a necklace. 20% wear a ring and 30% a necklace. Choosing a student at random, what is the probability that he/she wears:
   a) a ring or a necklace;
   b) a ring and a necklace.

(Answer: a) 40%; b) 10%)

(24) A total of 28% of American males smoke cigarettes, 7% cigars and 5% both cigarettes and cigars.
  a) What percentage of males smokes neither cigars nor cigarettes?
  b) What percentage of males smokes cigars but not cigarettes?

(Answer: a) 70%; b) 2%)

(25) If 8 rooks (castles) are randomly placed on a chessboard, compute the probability that none of the rooks can capture any of the others (no row or files contains more that one rook)?

(Answer: $\frac{7^2 \cdot 6^2 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2^2}{63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57}$)

(26) Two fair dice have both two of their sides painted red, two painted black, one yellow and one white. If the dice are rolled, what is the probability that both dice land with the same color face up?

(Answer: $\frac{5}{18}$)

(27) An urn contains 5 red, 6 blue and 8 green balls. If 3 balls are selected from the urn, what is the probability that the balls have
  a) the same color;
  b) three different colors.

(Answer: a) $\frac{86}{969}$; b) $\frac{240}{969}$)

(28) A forest contains 20 elks, of which 5 are captured, tagged and released. A certain time later, 4 of the 20 elks are captured, what is the probability that 2 of them have been tagged?

(Answer: $\frac{70}{323}$)

(29) An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student knows how to solve 7, what is the probability that at the final exam she knows how to solve
  a) every exercise
  b) at least 4.

(Answer: a) $\frac{1}{12}$; b) $\frac{1}{2}$)

(30) There are n socks, 3 of which are red, in a drawer. What is the value of n if, the probability of choosing a pair of red socks from the drawer is 1/2?

(Answer: $n = 4$)

(31) There are 5 hotels in a certain town. If 3 people check into hotels in a day, what is the probability that they each check into a different one? (Answer: $\frac{12}{25}$)

(32) A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men? (Answer: 0.4329)

(33) In a regular polygon with n edges 2 vertices are chosen at random. Compute the probability that these vertices are not consecutive. State the result in the case $n = 4, 5, 6$.

$\left(\frac{n-3}{n-1};\ \frac{1}{3};\ \frac{1}{2};\frac{3}{5}\right)$

(34) There are 30 physicists and 24 mathematicians attending a conference. A committee formed by 3 participants has to be chosen. If everyone is equally likely to take part of the committee, what is the probability that at least one mathematician takes part of it?

(Answer: 0.8363)

(35) A card is chosen at random from a deck of 52 cards. What is the probability that
a) the card is a club face card?
b) the card is either a club or a face?

(Answer: a) 3/52; b) 11/26)

(36) In a group of hundred people, 70 speak English, 45 Spanish, 23 both English and Spanish. Chosen a person at random, what is the probability that he/she does not speak English nor Spanish?

(Answer: 8%)

(37) Rolling a die, the faces 4 and 6 have the probability which is the double of the probability of the other faces. Describe the probability space and
a) Compute the probability that a odd number appears.
b) Compute the probability that the number is greater than 3.

(Answer: a) 3/8; b) 5/8)

(38) 3 dice are rolled and the lowest number obtained is recorded.What is the probability that : a) this number equals 6; b) is greater than 1; c) equals 1.

(Answer: a) $1/6^3$; b) $5^3/6^3$; c) $1 - 5^3/6^3$)

(39) If A and B are disjoint events then $\mathbb{P}(A) \leq \mathbb{P}(B^c)$: true or false?

(Answer: true)

(40) In a game with 2 dice "4 soft" means the sum of the points is 4 and one of the two dice is 1. What is the probability of getting a"4 soft"? and 4?

(Answer: 2/36; 3/36)

(41) A 4 digit code has to be selected as safe code. Compute the probability of:
(a) A="the 4 numbers are all different"
(b) B="the 4 numbers are strictly increasing"

(Answer: 0.504; 0.021)

(42) From a study the 10% of the students of LSE play an instrument in the leisure time, the 20% play some kind of sport, the 5% study a foreign language. Moreover, the 5% play an instrument and practice some sport, the 3% play an instrument and a foreign language and the 2% study a language and practice some sport; the 1% does the three things.. Choosing a student at random, what is the probability that he/she practices only sports? and that he/she studies a language and play an instruments but practise no sport?

(Answer: 14%; 2%)

(43) k indistinguishable particles randomly occupy k of n cells ($k \leq n$) (each cell contains at most one particle). If H is a subset formed by h cells. Compute the probability that all cells in H are occupied.

(Answer: $\frac{\binom{n-h}{k-h}}{\binom{n}{k}}$)

(44) From the set $E = \{e_1, \ldots, e_n\}$, k elements are chosen without replacement. Suppose $e_1, \ldots, e_m$ are red and $e_{m+1}, \ldots, e_n$ are white and let A_i be the event "i-th element of the k-chosen is red".

(a): Fix the sample space and S, compute $card(S)$.
(b): Compute $\mathbb{P}(A_i)$ and check it does not change with i.
(c): Compute $\mathbb{P}(A_i \cap A_j)$ and check it does not depend on i and j.

(Answer: $card(S) = \frac{n!}{(n-k)!}$; $\mathbb{P}(A_i) = \frac{m}{n}$; if $i = j$ then $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) = m/n$. If $i \neq j$ then $\mathbb{P}(A_i \cap A_j) = \frac{m(m-1)}{n(n-1)}$)

(45) (*) The game of bridge is played by 4 players, each of whom is dealt with 13 cards. How many bridge deals are possible?

(Answer: $\binom{52}{13,13,13,13} = \frac{52!}{13!13!13!13!}$)

(46) (*) If 12 people are to be divided into 3 committees of respective sizes 3, 4 and 5. How many divisions are possible?

(Answer: $\binom{12}{3,4,5} = \frac{12!}{3!4!5!}$)

(47) (*) If 8 new teachers are to be divided among 4 schools.

a) How many divisions are possible?
b) What if each school has to receive 2 teachers?

(Answer: a) 4^8; b) $\binom{8}{2,2,2,2} = \frac{8!}{2!2!2!2!}$)

(48) (*) 8 identical blackboards are to be divided among 4 schools.

a) How many divisions are possible?
b) What if each school must receive at least 1 blackboard?

(Answer: a) $\binom{11}{3} = 165$; b) $\binom{7}{3} = 35$)

(49) (*) 10 fishes are fished in a lake where 5 different species are present.

a) How many possible outcomes if an outcome is intended as the number of fishes for each of the species?
b) How many possible outcomes if 3 out of 10 fishes are trouts?
c) How many possible outcomes if at least 2 out of the 10 fishes are trouts?

(Answer: a) $\binom{14}{4} = 1001$; b) $\binom{10}{3} = 120$; c) $\binom{12}{4} = 495$)