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30401 MATHEMATICS AND STATISTICS - MODULE 2 (STATISTICS)

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EXERCISES ON DISCRETE RANDOM VARIABLES

(1) An exams consists of 5 multiple choice tests with 4 answers (only one of which is correct). By choosing the answer among the 4 randomly, compute a) the probability to answer correctly at least to four question; b) the average number of correct answers and the corresponding variance.

[a) $1/4^3$; b) 5/4; 15/16]

(2) Consider a particle starting at 0 that at each step is equally likely to move one position forward (+1) or backward (-1). a) Describe the position of the particle after two steps; b) what is the probability that after 50 steps the particle is at 25? and at 26? c) what is the probability that the particle moved the firs step forward given that at the step three is in -1?

[a) $p(-2) = p(2) = 1/4$, $p(0) = 1/2$; b) 0; $\binom{50}{38}\frac{1}{2^{50}}$; c) 1/3]

(3) We are looking for a key in a bunch of n keys. After choosing one at random, if it is not the right one, we mix it with the others and choose again from the bunch. a) What is the probability to find the key at the k-th trial? b) If two keys are needed to open the door. Compute the probability to open the door at the k-th trial?

[a) $\frac{1}{n}\left(1-\frac{1}{n}\right)^{k-1}$; b) $(k-1)\left(\frac{1}{n}\right)^2\left(1-\frac{1}{n}\right)^{k-2}$]

(4) In the national lotto 5 numbers out of 90 are selected weekly. There are people betting on the long overdue number (a number waited since many weeks). a) What is the probability that, betting on the long overdue number in a certain week, we win? b) If I keep on betting on it, in average after how many weeks do we expect to win for the first time?

[a) 5/90; b) 18 weeks]

(5) A newborn child independently inherites genes from his/her parents. Each of Eva's and Tom's children have O blood group with probability 0.25. Given they had 5 children and denote by X the number of children having O blood type (success). What kind of random variable is X? What is the probability that exactly two children have O? Which is the mean and standard deviation of X?

[$X \sim binom(5, 0.25)$; $\mathbb{P}(X = 2) = 0.2637$; $\mu = 1.25$; $\sigma = 0.9682$]

(6) An ordinary deck of 40 cards (20 red and 20 black) is shuffled and 10 cards are chosen (each card is equally likely to be red or black). Let X be the number of red cards chosen. a) Is X a binomial random variable? Compute $\mathbb{P}(X = 3)$.

b) If after choosing a card we always replace the card in the deck and shuffle it before choosing the next card, how would X be distributed? Which is the value of $\mathbb{P}(X = 3)$?

[a) no, there is no independence among trials and the probability of extracting red is changing each time: e.g., if the first card is black the second is more likely to be red, since there are more red cards in the deck; X is an hypergeometric. $\mathbb{P}(X = 3) = \frac{\binom{20}{3}\binom{20}{7}}{\binom{40}{10}} = 0.1042549$; b) $X \sim binom(10, 0.5)$; $\mathbb{P}(X = 3) = 0.1171875$]

(7) In a telephone survey it is known that the 20% of people contacted finish the questionnaire. Imagine the caller contact a sample of 15 people. Let X be the number of the ones who complete the survey. a) What distribution has X? what is the probability that exactly 3 complete the questionnaire? what the probability that at most 3 questionnaires have been completed? What is the expected number of filled questionnaires? and which is the standard deviation?
b) In a particular survey on bank accounts, the probability that a questionnaire is completed is only $p = 0.08$. What is the value of the standard deviation in this case? and with $p = 0.01$? c) What can we say about the variance when the probability of success approaches to 0? and 1?

[a) $X \sim binom(15, 0.2)$; $\mathbb{P}(X = 3) = 0.2501389$; $\mathbb{P}(X \leq 3) = 0.6481621$; $\mu = 3$; $\sigma = 1.549193$; b) $\sigma = 1.050714$ (se $p = 0.08$); $\sigma = 0.385357$ (se $p = 0.01$). c) Se $p \to 0$ either $p \to 1$, $\sigma = np(1 - p) \to 0$ i.e. the dispersion tends to zero and X converges to the mean value μ which for a binomial is $\mu = np$. Thus, if $p \to 0$, X tends to 0 (successes tend not to occur); if $p \to 1$, X tends to 15 (any trial tends to be a success)]

(8) A big supplier of electronic central units (ECU) has tested that the ECU that the 3% the whole production does not comply with the referred technical standard.
   (a) A motorbike producer received 200 ECU from the supplier.
      (i) what is the probability that all comply with the referred technical standards.
      (ii) What is the probability that at least 3 do not comply with the standards?
   (b) In 5 consecutive weeks the motorbike producer receives 200 ECU weekly. What is the probability that at least one week there is no ECU which does not comply with the standards out of the 200 supplied?

   [a)i) 0.002; ii) 0.94; b) 0.011]

(9) A wine glasses makers company produces two price category collections: Deluxe (D) and Standard (S). For both collections the packages contain 60 glasses each. Deluxe collection represents the 60% of the whole production, the standard represents the 40%. During the packaging process and shipping, the 2% of glasses from collection D is damaged. While the S collection has a percentage of damaged glasses which is 1%. A wine producer ordered a package of glasses.
   (a) If the glasses are from the D collection, what is the probability that at least one glass is damaged?
   (b) What is the previous probability it is not known the collection of the order?

(c) Suppose the package contains at least one damaged glass: what is the probability that the glasses belong to the deluxe collection?

[a) 0.7024 ; b) 0.6026; c) 0.6994]

(10) A driving practice test contains 20 multiple choice questions and the probability to get the wrong answer is 5% independently for each question. a) Compute the probability to give the wrong answer at least to a question. b) We complete 5 practice tests and we decide not to take the exam if in more than one practice test there is one or more wrong answers. Which is the probability that we do not sit the exam? c) If, instead, we do not sit the exam if out of all the 5 practice test multiple choice questions I give at least 5 wrong answers, what is the probability I will take the exam?

[a) 0.6415; b) 0.9411; c) 0.4360]

(11) Each question in an exam is composed of two parts, A and B. An answer to a question is correct if both the answers to A and B are correct. Let $p_A = 0.15$ ($p_B = 0.20$) be the probability of the event the student gives a wrong answer to part A (B), respectively, and suppose the two events are independent.

(a) Chosen a question at random, compute the probability p that students answer correctly?

(b) Write the FORMULA for the probability that in a test with 100 questions (each composed of the two parts A and B), at least 80 student's answers are correct.

[a) 0.68; b) $\mathbb{P}(X \geq 80) = \sum_{k=80}^{100} \binom{100}{k} (0.68)^k (0.32)^{100-k}$]

(12) Two surgeons (Black and Fisher) are in charge of heart surgeries at a certain hospital. Black is usually doing 50 surgeries a year and his percentage of unsuccessful ones is 3%. The second is doing 100 surgeries each year and the percentage of unsuccessful ones is 1%.

(a) Find the probability that at that hospital a heart surgery will be unsuccessful.

(b) Find the probability that an unsuccessful surgery has been done by Black.

(c) In average after how many surgeries the first unsuccessful occurs?

[a) 1/60; b) 3/5; c) 60]

(13) A department store sells potato chips bags, each containing a bonus with probability 0.4. A buyer receives a free sticker each 4 bonus and if collects 7 bonus receives additionally a free potato chips bag. Asher bought 10 potato chips bags. Let X be the random variable representing the number of bonus Asher found and compute: a) the mass probability function of X; b) the probability that Asher receives the sticker but not the free chips;c) the probability that Asher does not receive anything free.

[a) $X \sim binom(10, 0.4)$; b) 0.5629; c) 0.3823]

(14) According to the Census Bureau data, the 9.96% of the American adults (18 years old or older) are Hispanic. In a social survey a random sample of 1200 adults has been interviewed. a) What is the average number if Hispanic people interviewed? b) What is the probability that in the random sample there are less than 100 Hispanic people?

$\left[\text{a) } 119.52\text{; b) } \mathbb{P}(X < 100) = \sum_{k=0}^{99} \binom{1200}{k} (0.0996)^k (0.9004)^{1200-k}\right]$

(15) An engineering system consisting of $n = 10$ components is said to be a 6-out-of-10 system if the system functions if at least 6 of the 10 components function. If all components function independently of each other and each functions with probability $p = 1/2$, compute:
 (a) the probability the system does not function;
 (b) the probability that a component does not function knowing that all the others do function;
 (c) the average number of not functioning components.

[a) 0.6230 ; b) 1/2; c) 5]

(16) Typically 1 or 2 star table tennis balls are bought in bulk (packs of 100) and used for rallying or practice. The probability that a ball is damaged is 0.01, independently of the others. If there are more than 3 damaged balls in a bulk, the entire bulk is replaced by a new one.
 (a) What is the probability that a bulk is replaced?
 (b) Which is the average number of defective balls in a bulk?
 (c) If a department store orders a bulk weekly, what is the probability the first bulk to be replaced occurs in the first month?
 (d) In average after how many weeks the first bulk is asked to be replaced.

[a) 0.0184 ; b) 1; c) 0.0715; d) 55 (approx. with the first bigger integer)]

(17) A lake contains N fishes where N is unknown. We capture and mark 100 fishes an let them free. A few days later, we capture 50 fishes and find that k are marked. Find the value of N which maximizes the probability $P(X = k)$ with respect to N (the maximum likelihood estimate of N),
 (a) if $k = 35$,
 (b) If $k = 5$.

[a) $\hat{N} = \lfloor 100 \cdot 50/35 \rfloor = 142$; b) $\hat{N} = \lfloor 100 \cdot 50/5 \rfloor = 1000$, where $\lfloor x \rfloor$ denotes the integer part of x].