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30401 MATHEMATICS AND STATISTICS - MODULE 2 (STATISTICS)

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EXERCISES ON JOINTLY DISTRIBUTED RANDOM VARIABLES

I. MULTIPLE CHOICE TEST: *Select the correct answer.*

(1) X and Y are two uncorrelated standard normal random variables and $Z = \frac{X+Y}{2}$. Then

$\boxed{\text{A}}$ $\mathbb{P}[\,|Z| < 1\,] < \mathbb{P}[\,|X| < 1\,]$

$\boxed{\text{B}}$ $\mathbb{P}[\,|Z| < 1\,] = \mathbb{P}[\,|X| < 1\,]$

$\boxed{\text{C}}$ $\mathbb{E}[\,2Z^2\,] = 0$

$\boxed{\text{D}}$ $\mathbb{E}[\,2Z^2\,] = 1$

(2) X_1 is a uniform random variable on $[0,1] \subseteq \mathbb{R}$ and X_2 a uniform random variable on $[0,2] \subseteq \mathbb{R}$. If X_1 and X_2 are independent then $\mathbb{P}[X_1 > X_2]$ is equal to

$\boxed{\text{A}}$ $1/8$

$\boxed{\text{B}}$ $1/2$

$\boxed{\text{C}}$ 0

$\boxed{\text{D}}$ $1/4$

(3) Let X_1 and X_2 be two normal random variables with the same mean μ and the same variance $\sigma^2 > 0$, with correlation $\rho = -1/2$. Then

$\boxed{\text{A}}$ $Var(X_1 + X_2) \geq 2\sigma^2$

$\boxed{\text{B}}$ $Var(X_1 + X_2) \leq \sigma^2$

$\boxed{\text{C}}$ $\mathbb{E}(X_1 + X_2) < 2\mu$

$\boxed{\text{D}}$ $\mathbb{E}(X_1 + X_2) > 2\mu$

(4) X and Y are a couple of jointly continuous random variable with density function $f_((x,y) = ke^{-2x-y}$ if $x > 0, y > 0$ and null otherwise. Then

$\boxed{\text{A}}$ $k = 0$

$\boxed{\text{B}}$ $k = \frac{1}{2}$

$\boxed{\text{C}}$ $k = 2$

$\boxed{\text{D}}$ $k = 3$

(5) X and Y are two independent standard normal random variable. Then

$\boxed{\text{A}}$ $\mathbb{P}[4X - 3Y > 1] = 0.8413$

$\boxed{\text{B}}$ $\mathbb{P}[4X + 3Y < -1] = 0.7734$

$\boxed{\text{C}}$ $\mathbb{P}[3X - 4Y > 1] = 0.1587$

$\boxed{\text{D}}$ $\mathbb{P}[3X + 4Y < -1] = 0.4207$

(6) X and Y are two independent binomial random variables of parameters $n = 3$, $p = 0.2$ and $n = 7$, $p = 0.2$, respectively. Then

$\boxed{\text{A}}$ $\mathbb{P}[X + Y = 3] = \binom{10}{3}(0.2)^3$ $\boxed{\text{B}}$ $\mathbb{P}[X + Y = 3] = \binom{7}{3}(0.2)^3(0.8)^7$

$\boxed{\text{C}}$ $\mathbb{P}[X + Y = 3] = \binom{10}{3}(0.2)^3(0.8)^7$ $\boxed{\text{D}}$ $\mathbb{P}[X + Y = 3] = 1$

(7) X and Y are two uncorrelated Bernoulli random variables with parameters $\frac{1}{3}$ and $\frac{2}{3}$, respectively.

$\boxed{\text{A}}$ $\mathbb{P}[XY = 0] = \frac{1}{3}$ $\boxed{\text{B}}$ $\mathbb{E}[XY] = \frac{1}{9}$

$\boxed{\text{C}}$ $\mathbb{P}[XY = 1] = \frac{2}{9}$ $\boxed{\text{D}}$ $\mathbb{E}[X - Y] = 0$

(8) X and Y are two Gaussian random variables, respectively $X \sim \mathcal{N}(1, 9)$ and $Y \sim \mathcal{N}(2, 4)$, with correlation $\varrho(X, Y) = -\frac{1}{2}$. Then

$\boxed{\text{A}}$ $Var(X - Y) = 7$ $\boxed{\text{B}}$ $Var(X - Y) = 5$

$\boxed{\text{C}}$ $Var(X - Y) = 13$ $\boxed{\text{D}}$ $Var(X - Y) = 19$

(9) Let X_1 and X_2 be two independent random variables with same mean equal to 1 and same variance equal to 4. Then

$\boxed{\text{A}}$ $\mathbb{E}[(X_1 + X_2)^2] = 12$ $\boxed{\text{B}}$ $\mathbb{E}[(X_1 + X_2)^2] = 10$

$\boxed{\text{C}}$ $\mathbb{E}[(X_1 + X_2)^2] = 4$ $\boxed{\text{D}}$ $\mathbb{E}[(X_1 + X_2)^2] = 8$

(10) X_1 and X_2 are two independent random variables that take on the values 0 or 2, both with probability 1/2. Then

$\boxed{\text{A}}$ $\mathbb{E}[\min\{X_1, X_2\}] = 1/2$ $\boxed{\text{B}}$ $\mathbb{E}[X_1(X_2 + 1)] = 1$

$\boxed{\text{C}}$ $\mathbb{E}[\min\{X_1, X_2\}] = 0$ $\boxed{\text{D}}$ $\mathbb{E}[X_1(X_2 + 1)] = 3$

(11) X is an exponential random variable of parameter 2 and Y a Poisson random variable of parameter 2. Suppose X and Y are independent. Then

$\boxed{\text{A}}$ $\mathbb{E}[4X^2 + 3Y^2] = 20$ $\boxed{\text{B}}$ $Var(4X + 3Y - 1) = 6$

$\boxed{\text{C}}$ $\mathbb{E}[4X^2 + 3Y^2] = 13$ $\boxed{\text{D}}$ $Var(4X + 3Y - 1) = 21$

(12) X is a normal random variable with mean 1 and variance 4 and Y a Bernoulli random variable of parameter $\frac{1}{3}$. Suppose X and Y are independent. Then

$\boxed{\text{A}}$ $\mathbb{P}[2XY > -1] = 0.7734$ $\boxed{\text{B}}$ $\mathbb{P}[2XY > 1] = 0.1996$

$\boxed{\text{C}}$ $\mathbb{P}[2XY > 1] = 0.5987$ $\boxed{\text{D}}$ $\mathbb{P}[2XY > -1] = 0.2578$

(13) Let X and Y be two independent Bernoulli random variables of parameter $\frac{1}{3}$. Consider $U = X \cdot Y$ and $V = X + Y$. Then

$\boxed{\text{A}}$ $Cov(U, V) = \frac{2}{9}$ $\boxed{\text{B}}$ $Cov(U, V) = 0$

$\boxed{\text{C}}$ $\mathbb{P}(V = 0|U = 0) = 1$ $\boxed{\text{D}}$ $\mathbb{P}(V = 0|U = 0) = \frac{1}{2}$

(14) Let $X_1, X_2, \ldots X_n$ be a random sample from a normal with parameters $\mu = 1$ and $\sigma^2 = 1$ and denote by $\bar{X}$ the sample mean. Then

$\boxed{\text{A}}$ $Var(\bar{X}) = 1$ $\boxed{\text{B}}$ $Var(X_1 - \bar{X}) = 1 - \frac{1}{n}$

$\boxed{\text{C}}$ $Var(X_1 - \bar{X}) = 1 + \frac{1}{n}$ $\boxed{\text{D}}$ $Var(\bar{X}) = \frac{1}{n^2}$

(15) Let $X_1, X_2, \ldots X_5$ be a random sample from a normal with parameters $\mu = 1$ and $\sigma^2 = 5$ and denote by $\bar{X}$ the sample mean. Then

$\boxed{\text{A}}$ $Var(\bar{X}) = 1$ $\boxed{\text{B}}$ $\mathbb{P}(|\bar{X} - 1| < 0.5) = 0.0796$

$\boxed{\text{C}}$ $Var(\bar{X}) = \frac{1}{5}$ $\boxed{\text{D}}$ $\mathbb{P}(|\bar{X} - 1| < 0.1) = 0.5398$

(16) An urn contains 5 balls numbered from 1 to 5, the balls are selected without replacement and we denote $X_i = 1$ if the i-th ball selected is the ball number i and $X_i = 0$ otherwise. Then

$\boxed{\text{A}}$ $Var(X_1 + X_2) = \frac{8}{5}$ $\boxed{\text{B}}$ $Cov(X_1, X_2) = -0.01$

$\boxed{\text{C}}$ $Var(X_1 - X_2) = 0$ $\boxed{\text{D}}$ $Cov(X_1, X_2) = 0.01$

(17) From an urn containing 7 black balls and 3 white, 3 balls are selected without replacement. Let $X_i = 1$ if ball i-th ball is black and $X_i = 0$ otherwise, then

$\boxed{\text{A}}$ $Cov(X_1, X_2) = 0$ $\boxed{\text{B}}$ $\mathbb{E}(X_1 + X_2 + X_3) = 2.1$

$\boxed{\text{C}}$ $Var(X_1 + X_2 + X_3) = 0.63$ $\boxed{\text{D}}$ $Cov(X_1, X_3) = 0.21$

(Answers: 1) D; 2) D; 3) B; 4) C; 5) D; 6) C; 7) C; 8) D; 9) A; 10) A; 11) A; 12) B; 13) D; 14) B; 15 A; 16) D ;17) B.)

— II. EXERCISES —

(1) Let X and Y two random variables with the following joint distribution:

$X\backslash Y$	0	1
-1	$1/8$	$1/8$
0	$3/16$	$5/16$
1	$1/8$	$1/8$

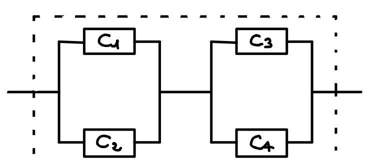
(a) Find the marginal probability mass functions of X and Y. Are X and Y independent?
(b) Find the joint distribution function $F(x, y)$ at $(0, 0)$ and $(-1, \frac{1}{2})$.
(c) Find the m.p.f. of X conditional to $Y = 1$.
(d) Find expectation and variance of $X, Y, X + Y$.

$\big($Answer: a) m.p.f. of X: $p(-1) = p(1) = \frac{1}{4}; p(0) = \frac{1}{2}$ and m.p.f. of Y: $p(0) = \frac{7}{16}; p(1) = \frac{9}{16}$; not independent; b) $F(0, 0) = \frac{5}{16}$ and $F(-1, \frac{1}{2}) = \frac{1}{8}$; c) $p_{X|Y}(-1|1) = p_{X|Y}(1|1) = \frac{2}{9}; p_{X|Y}(0|1) = \frac{5}{9}$; d) $\mathbb{E}(X) = 0; \mathbb{E}(Y) = \mathbb{E}(X + Y) = \frac{9}{16}; Var(X) = \frac{1}{2}; Var(Y) = \frac{63}{256}; Var(X + Y) = \frac{191}{256}\big)$

(2) The time that one train takes from station A to B follows a normal distribution with mean 60 (minutes) and standard deviation 12. The time specified in the train time table is 1h and 15 mins. Assume that the train leaves A station on time.
(a) What is the probability that the train arrives more than 5 minutes late at station B?
(b) Each day there are 50 trains from A to B. Which is the average number of trains which arrive more than 5 minutes late in a day?
(c) What time should be specified in the train time table so that the probability that the train arrives late at B is equal to 0.0062?

$\big($Answer: a) 0.0475; b) 2.375 ; c) 90 mins (1h and 30 mins) $\big)$

(3) An electronic device is formed by 4 components $C_1 ... C_4$, as shown in the picture, whose lifetimes are i.i.d. random variables $T_i \sim Exp(1), \forall i = 1...4$.



(a) Observe that each component functions in $t = 1$ with probability $p = e^{-1}$ and compute the probability that the device functions in $t = 1$.
(b) Find the survival function of the device lifetime.
(c) Find the average lifetime of the device.

$\big($Answer: a) 0.3605; b) $R(t) = e^{-2t}(2 - e^{-t})^2$, if $t \geq 0$ and 1 otherwise; c) $\frac{11}{12}$ $\big)$

(4) The number of people that enter in a drugstore in a given hour is a Poisson random variable with parameter $\lambda = 10$. Compute
(a) the probability that in the given hour enter at most 3 people;
(b) the probability that the first person enters after the first half an hour (thus in the first half hour nobody enters).

(c) Suppose that the percentage of women entering in the drugstore (in that hour) is 20%, find the probability that given 10 persons entered in the drugstore 3 are women.
(d) Find the distribution of the number of women who enter enter in a drugstore in the given hour.

(Answer: a) 0.0103; b) 0.0067; c) 0.2013; d) Poisson(2))

(5) Let X and Y be two discrete random variables both taking values $0, 1, 2$. Suppose that their joint probability mass function is:

$$p_{XY}(x,y) = \begin{cases} 0.1 & \text{if} \quad (x,y) \in \{(0,0),(1,1),(1,2),(2,0)\} \\ 0.2 & \text{if} \quad (x,y) \in \{(0,1),(0,2),(2,1)\} \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal probability mass functions of X and Y.
(b) Are the two random variables X and Y independent?
(c) Compute $\mathbb{P}((X \le 1) \cap (Y \le 1))$ and $\mathbb{P}((X \le 1) \cup (Y \le 1))$.
(d) Compute $\mathbb{P}\left(X < \frac{1}{2} | Y > 1)\right)$ and the expectation of X given $Y = 2$.

(Answer: a) m.p.f. of X: $p(0) = 0.5; p(1) = 0.2; p(2) = 0.3$and m.p.f.. of Y: $p(0) = 0.2; p(1) = 0.5; p(2) = 0.3$; b) b) not independent; c) 0.4 and 1; d) $\frac{2}{3}$ and $\frac{1}{3}$)

(6) Let X and Y be two discrete random variables with joint probability mass function represented in the table below

$X \backslash Y$	0	1
-1	0	p
0	p	0
1	1/6	1/6

Compute:
(a) p;
(b) the marginal probability mass functions of X and Y. Are X and Y independent?
(c) the expectation of X;
(d) the expectation and the variance of Y;
(e) the expectation of Y given $X = 0$.
(f) $\mathbb{P}(U = 0)$, where $U = XY$.

(Answer: a) $p = \frac{1}{3}$; b) m.p.f. of X: $p(-1) = p(0) = p(1) = \frac{1}{3}$;and m.p.f.. of Y: $p(0) = p(1) = 0.5$; not independent; c) 0; d) 0.5 and 0.25; e) 0; f) 0.5)

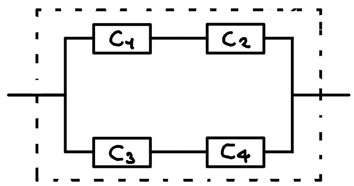
(7) Today we invest 100 Euro in a financial asset. Each day the value of the investment can reduce to the half of its value with probability 1/6, or remain of the same value with probability 1/2, or double its value with probability 1/3, each time independently of what happened previously.
Denote by S_n the value of the investment in the day n-th (with $S_0 = 100$) and note that $S_n = S_0 * X_1 * X_2 * \cdots * X_n$ where X_i are independent identically distributed random variables which take on the values 1/2, 1 and 2 with probabilities 1/6, 1/2 and 1/3, respectively.
Compute:
(a) the mass probability function of S_2 the value of the investment in the second day and its expectation;

(b) the covariance of the couple (S_1, S_2) and say if S_1 and S_2 are positively or negatively correlated;

(c) the probability that $S_1 = 100$, given $S_2 = 100$.

$\big($Answer: a) $p(25) = \frac{1}{36}; p(50) = \frac{1}{6}; p(100) = \frac{13}{36}; p(200) = \frac{1}{3}; p(400) = \frac{1}{9}$; 156.25; b) $\frac{5^6}{4}$; positively correlated; c) $\frac{9}{13}\big)$

(8) An electronic device is formed by 4 components $C_1 ... C_4$, as shown in the picture, whose lifetimes are i.i.d. random variables $T_i \sim Exp(1), \forall i = 1...4$.



(a) Observe that each component functions in $t = 2$ with probability $p = e^{-2}$ and compute the probability that the device does NOT function in $t = 2$.

(b) Find the survival function of the device lifetime.

(c) Find the average lifetime of the device.

$\big($Answer: a) 0.9637; b) $R(t) = e^{-2t}(2 - e^{-2t})$, if $t \geq 0$ and 1 otherwise; c) $\frac{3}{4}\big)$

(9) The batteries of a certain company are produced by two factories. The lifetimes of the batteries are independent exponentially distributed random variables with mean 120 hours if they are produced by factory A and mean 90 if produced by factory B. We have batteries of this company and each one, independently, is equally likely to be produced by factory A or B. Suppose we pick one:

(a) Compute the probability that the battery lasts less than 60 hours.

(b) Compute the probability that the battery has been produced by factory A, given it lasted less than 60 hours.

Suppose now we pick two of our batteries. Compute:

(c) the probability that both last less than 60 hours, given they both have been produced by A;

(d) the probability that both last less than 60 hours.

$\big($Answer: a) 0.44; b) 0.458 ; c) 0.1548; d) $(0.44)^2\big)$

(10) In an urn there are 8 dice. The half of them are fair, while the other half are biased, in such a way that each biased die lands showing face 2 with probability $\frac{1}{2}$ and the other faces with equal probability. A die is chosen at random from the urn and it is thrown several times. Let X_i be the face shown in the i-th throw. Compute

(a) the probability that $X_1 = 2$;

(b) the average of X_1;

(c) the probability that $X_1 = 2$ and $X_2 = 1$. Are X_1 and X_2 independent?

(d) the probability that the selected die is fair, given that $X_1 = 2$ and $X_2 = 1$.

$\big($Answer: a) $\frac{1}{3}$; b) $\frac{16}{5}$; c) $\frac{7}{180}$; d) $\frac{5}{14}\big)$