STATISTICS

Module 2 *of*
MATHEMATICS AND STATISTICS

Marina Santacroce

marina.santacroce@unibocconi.it
Department of Decision Sciences, Bocconi University

L10. Jointly distributed random variables

Outline

Jointly distributed random variables
- Joint probability distribution functions
- Joint probability mass function
- Joint probability density function

Independent random variables
- Independent random variables
- Sums of independent random variables

Conditional distribution
- Conditional distribution and conditional expected value

Introduction

Thus far we have concerned with probability distributions for single random variables.
However we are often interested in probability statements involving two or more random variables.

- In the roll of three dice we are interested in the minimum number appeared on the dice;
- If I don't want to wait more than 10 minutes at the bus stop and I can catch both the buses 3 and 45;
- If a system is formed by n components, their lifetimes will determine the lifetime of the system;
- If I need a first year average grade not less than 28, I will be concerned with all the first year grades...

In all these examples we need to study more random variables jointly.

Bivariate probability distribution function

Assume we have two real random variables X and Y.

Def. We define the *cumulative probability distribution function* of X and Y $F_{XY} : \mathbb{R}^2 \to [0;1]$ by:

$$F_{XY}(x,y) = \mathbb{P}(X \leq x, Y \leq y) \qquad x, y \in \mathbb{R}.$$

Remark: The distributions of X and Y can be obtained from the F_{XY} as follows:

$$F_X(x) = \lim_{y \to +\infty} F(x,y) \qquad F_Y(y) = \lim_{x \to +\infty} F(x,y)$$

We refer to F_X and F_Y as the *marginal* distributions of X and Y.

- Given F_{XY}, one can compute all *joint probability statements about X and Y*, e.g.

$$\mathbb{P}(x_1 < X \leq x_2, y_1 < Y \leq y_2) = F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) + F(x_1,y_1)$$

Joint probability mass function

Let X and Y be two discrete random variables.

Def. We define the *joint probability mass function* of X and Y
$p : \mathbb{X} \times \mathbb{Y} \to [0, 1]$

$$p(x, y) = \mathbb{P}(X = x, Y = y) \qquad (x, y) \in \mathbb{X} \times \mathbb{Y}^1.$$

$$\Rightarrow \quad p(x, y) \geq 0 \text{ for any } (x, y) \quad \text{and} \quad \sum_{(x,y)\in\mathbb{X}\times\mathbb{Y}} p(x, y) = 1.$$

The *marginal probability mass functions of X and Y* are respectively:

$$p_X(x) = \sum_{y\in\mathbb{Y}} p(x, y) \quad \text{with} \quad x \in \mathbb{X},$$
$$p_Y(y) = \sum_{x\in\mathbb{X}} p(x, y) \quad \text{with} \quad y \in \mathbb{Y}.$$

$$\triangleright \qquad p_X(x) = \mathbb{P}(X = x) = \mathbb{P}(X = x, \bigcup_{y\in\mathbb{Y}} (Y = y)) =$$
$$\mathbb{P}(\bigcup_{y\in\mathbb{Y}} (X = x, Y = y)) = \sum_{y\in\mathbb{Y}} p(x, y)$$

---

[1] $\mathbb{X}$ and $\mathbb{Y}$ are, respectively, the set of values assumed by X and Y

Example

Suppose that **3 balls** are randomly selected from an urn containing 3 red, 4 blue and 5 white balls. Let X and Y be respectively the number of red balls and blue balls drawn. Find the **joint mass probability function of X and Y**.

▷ X and Y can take on the values $\{0, 1, 2, 3\}$.

The joint mass function:

$p(i, j) = \mathbb{P}(X = i, Y = j)$ for any i and $j \in \{0, 1, 2, 3\}$ such that $i + j \leq 3$.

$p(0, 0) = \mathbb{P}(\{3 \text{ white balls drawn}\}) = \frac{\binom{5}{3}}{\binom{12}{3}}$

$p(0, 1) = \mathbb{P}(\{1 \text{ blue and } 2 \text{ white balls drawn}\}) = \frac{\binom{4}{1}\binom{5}{2}}{\binom{12}{3}}$.

$\Rightarrow$ In general,

$$p(i, j) = \frac{\binom{3}{i}\binom{4}{j}\binom{5}{3-(i+j)}}{\binom{12}{3}}$$

◇ Verify that $p_X(1) = \sum_{j:0}^{3} p(1, j)$

Note that $p_X(1) = \mathbb{P}(\{1 \text{ red ball drawn}\}) = \frac{\binom{3}{1}\binom{9}{2}}{\binom{12}{3}}$

Joint probability density function

Def. Two random variables X and Y are said *jointly* (absolutely) *continuous* if there exists a function f, named *joint probability density function* $f : \mathbb{R} \times \mathbb{R} \to [0; +\infty)$ such that, whenever the integral is well defined,

$$\mathbb{P}((X, Y) \in C) = \iint_C f(x, y) dxdy.$$

Thus, f satisfies

$$\int_{\mathbb{R}^2} f(x, y) dxdy = 1.$$

- In particular we have

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) dudv$$

- thus, differentiating

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y).$$

- If dx and dy are “small”

$$\mathbb{P}(x \le X < x+dx, y \le Y < y+dy) = \int_{x}^{x+dx} \int_{y}^{y+dy} f(u, v) dudv \approx f(x, y) dxdy.$$

Example: System failure

Consider the function $f(x, y) = \begin{cases} kx & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

where k is a constant. Find k so that f is a density function of two random variables.

- A device contains two components. The device fails if either component fails. The joint probability density function of the lifetimes of the components, measured in hours, is $f(x, y)$ with $0 < x < 1$ and $0 < y < 1$.
  What is the probability that the device fails during the first half hour of operation?
- A device runs until at least one component runs. When both the components fail the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is

  $$f(x, y) = 2e^{-x}e^{2y} \qquad \text{for } x > 0 \text{ and } y > 0$$

  and null otherwise. What is the probability that the device fails during its first hour of operation?

Marginal densities

If X and Y are two jointly absolutely continuous random variables, then the two random variables X and Y are continuous.

- The density function of X is said marginal density function of X

$$f_X(x) = \int_{\mathbb{R}} f(x,y)dy, \quad x \in \mathbb{R}$$

$\Rightarrow$ it is obtained by differentiating

$$F_X(x) = \mathbb{P}(X \leq x, -\infty < Y < +\infty) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f(u,y)dydu$$

with respect to x.

- Similarly, one finds the marginal density function of Y

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)dx, \quad y \in \mathbb{R}$$

Example:

The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < +\infty \; 0 < y < +\infty \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal distribution function of X and its marginal density function.

Independent random variables

In probability *independent random variables* play a fundamental rule.

The definition of independence between random variables is based on the concept of *independent events*.

Def. Two random variables are said **independent** if the events written in terms of X and the events written in terms of Y are independent, i.e.

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B) \quad \text{for any } A \text{ and } B$$

Loosely speaking, knowing the value of one, say X, does not change the distribution of the other, Y.

It can be shown that X and Y are **independent** if and only if:

$F_{XY}(x, y) = F_X(x)F_Y(y)$ for any x and $y \in \mathbb{R}$.

Moreover one can show that the definition of independence can be given also in terms of the probability mass functions/density functions as follows

$$p_{XY}(x, y) = p_X(x)p_Y(y) \text{ for any } x \in \mathbb{X} \text{ and } y \in \mathbb{Y},$$

if the random variables are *discrete* and, analogously, in the *continuous case* as

$$f_{XY}(x, y) = f_X(x)f_Y(y) \text{ for any } (x, y) \in \mathbb{R}^2.$$

Independent random variables: examples

1. Ugo and Zoe decide to meet in front of the cinema. They agree they will wait each other outside not more than 10 minutes. If each of them independently arrives at a time that is uniformly distributed between 21.00 and 22.00, what is the probability that they will enter together in the cinema?
2. The joint density function of X and Y is

$$f(x,y) = 6e^{-2x}e^{-3y} \qquad \text{if } x > 0 \text{ and } y > 0$$

equals 0 otherwise. Are X and Y independent?

if the density is

$$f(x,y) = 24xy, \qquad \text{if } 0 < x < 1,\ 0 < y < 1,\ 0 < x+y < 1,$$

and null otherwise, are X and Y independent?

Sums of independent random variables

- Assume X and Y are two *independent* random variables. We want to find the distribution of $Z = X + Y$.

Discrete case If X and Y are two discrete random variables taking on values in the *integers* of mpf p_X and p_Y respectively, then for any z integer

$$\begin{aligned} p_{X+Y}(z) &= \mathbb{P}(X+Y=z) \\ &= \sum_y \mathbb{P}(X+Y=z, Y=y) \\ &= \sum_y \mathbb{P}(X+y=z, Y=y) \\ &= \sum_y \mathbb{P}(X=z-y)\mathbb{P}(Y=y) \\ &= \boxed{\sum_y p_X(z-y)p_Y(y) = p_X \star p_Y(z)}^{2} \end{aligned}$$

[2] this operation is said *convolution*.

Convolution: continuous case

Continuous Case If X and Y are two random variables with density f_X and f_Y respectively, the distribution of $X+Y$ is said *convolution* of the distributions of X and Y and its density is

$$\boxed{f_Z(z) = f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y) dy = f_X \star f_Y(z)}$$ [3]

[3] this operation is said *convolution*.

Convolutions: examples

We recall the following well known cases of convolutions:

- $\text{Binom}(n, p) \star \text{Binom}(m, p) = \text{Binom}(n + m, p)$
- $\text{Poisson}(\lambda_1) \star \text{Poisson}(\lambda_2) = \text{Poisson}(\lambda_1 + \lambda_2)$
- $\text{Exp}(\lambda) \star \text{Exp}(\lambda) = \text{Gamma}(2, \lambda)$
- $\text{N}(\mu_1, \sigma_1^2) \star \text{N}(\mu_2, \sigma_2^2) = \text{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Conditional mass probability function and conditional expected value

Def. Let X and Y be two discrete random variables with joint mass probability function $p(x, y)$. For any y such that $p_Y(y) > 0$, we define the function

$$\boxed{p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)}, \; x \in \mathbb{X}.}$$

Prop. The function is said conditional mass probability function of X given $Y = y$, $p_{X|Y}(\cdot|y)$, and is indeed a *mass probability function* for X since

$$p_{X|Y}(x|y) \geq 0 \qquad \text{and} \qquad \sum_{x \in \mathbb{X}} p_{X|Y}(x|y) = 1.$$

Rem. Note that given p_Y and $p_{X|Y}$, the joint mass probability function of X and Y is given by:

$$\boxed{p_{XY}(x, y) = p_{X|Y}(x|y) p_Y(y)}$$

Def. The expected value of X given $Y = y$ is

$$\mathbb{E}(X|Y = y) = \sum_x x \, p_{X|Y}(x|y)$$

Conditional density and expected value: continuous case

Def. If X and Y are jointly continuous with joint density function $f(x, y)$, for any y such that $f_Y(y) \neq 0$ the *conditional density of X given* $Y = y$

$$\boxed{f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} \quad x \in \mathbb{R}}$$

- One can easily check that $f_{X|Y}(\cdot|y)$ is a density function for X.
- Note that

$$\boxed{f_{XY}(x, y) = f_{X|Y}(x|y) f_Y(y)}$$

Def. The expected value of X given $Y = y$ is

$$\mathbb{E}(X|Y = y) = \int_{-\infty}^{+\infty} x \, f_{X|Y}(x|y) \, dx.$$

We remark it is function of y.

Prop. Given two random variables X and Y, verify that

$$\boxed{\mathbb{E}(X) = \mathbb{E}\left(\mathbb{E}(X|Y)\right).}$$

◇ Denote by $g(Y) = \mathbb{E}(X|Y)$ and compute $E(g(Y))$.