STATISTICS

Module 2 *of*
MATHEMATICS AND STATISTICS

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L13: Distributions arising from the normal: Chi square, t and F

References: S.Ross *Introduction to* Probability and Statistics for Engineers and Scientists: Section 5.8: 5.8.1, 5.8.2 and 5.8.3

Outline

Gaussian or normal random variable

- A continuous random variable X is said normal or Gaussian, $X \sim \mathrm{N}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$, if the density function is of the form

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.}$$

◇ $\boxed{\mathbb{E}(X) = \mu}$ and $\boxed{\mathrm{Var}(X) = \sigma^2}$.

- If $X \sim \mathrm{N}(0, 1)$ the random variable is said standard normal and the density function is:

$$\boxed{f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad z \in \mathbb{R}.}$$

- Recall that if $X \sim \mathrm{N}(\mu, \sigma^2)$, $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable.

Distribution of the square of a standard normal

- Consider a standard normal random variable $Z \sim \mathrm{N}(0,1)$.
- Compute the distribution function and the density function of the *square* of the *standard normal* Z^2

$$F_{Z^2}(u) = \mathbb{P}(Z^2 \le u) = \mathbb{P}(|Z| \le \sqrt{u}) = \begin{cases} 0 & u \le 0 \\ 2F_Z(\sqrt{u}) - 1 & u > 0 \end{cases}$$

Differentiating the distribution function we obtain the density function

$$f_{Z^2}(u) = F'_{Z^2}(u) = \begin{cases} 2f_Z(\sqrt{u})\frac{1}{2\sqrt{u}} = \frac{e^{-\frac{u}{2}}(\frac{1}{2})^{\frac{1}{2}}u^{-\frac{1}{2}}}{\sqrt{\pi}}, & u > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this density with the one of a gamma(α, λ)

$$f(x) = \begin{cases} \frac{\lambda^{\alpha} e^{-\lambda x} x^{\alpha - 1}}{\Gamma(\alpha)} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

we conclude that the square of the standard normal is a gamma$(\frac{1}{2}, \frac{1}{2})$.

- Recall $\Gamma(\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$ and check the value of the constant $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

MGF of the square of a normal random variable

- Consider a standard normal random variable $Z \sim \mathrm{N}(0,1)$.
- Compare the distribution obtained by computing the MGF of Z^2.

The MGF of Z^2 is

$$\phi_{Z^2}(t) = \mathbb{E}\left(e^{tZ^2}\right) = \left(\frac{1/2}{1/2 - t}\right)^{\frac{1}{2}}, \quad t < \frac{1}{2}$$

and, recalling the MGF of $X \sim \text{gamma}(\alpha, \lambda)$,

$$\phi_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}, \qquad t < \lambda$$

$$\Rightarrow \qquad Z^2 \sim \text{gamma}\left(\frac{1}{2}, \frac{1}{2}\right).$$

Sum of the squares of i.i.d. standard normals and Chi square

- Consider n independent standard normal random variables $Z_1, Z_2, \ldots Z_n$,
- The sum of the squares $Z_1^2 + Z_2^2 + \ldots Z_n^2$ is known as Chi square χ_n^2 with n degrees of freedom.
- The distribution of the χ_n^2 with n degrees of freedom is a gamma$(\frac{n}{2}, \frac{1}{2})$.

Recall the property of the MGF for sum of independent random variables

$$\phi_{\chi_n^2} = \phi_{Z_1^2+Z_2^2+\ldots Z_n^2}(t) = \mathbb{E}\left(e^{t(Z_1^2+\ldots Z_n^2)}\right) = \prod_{i:1}^{n}\mathbb{E}\left(e^{tZ_i^2}\right) = \left(\frac{1/2}{1/2-t}\right)^{\frac{n}{2}}, \quad t < \frac{1}{2}$$

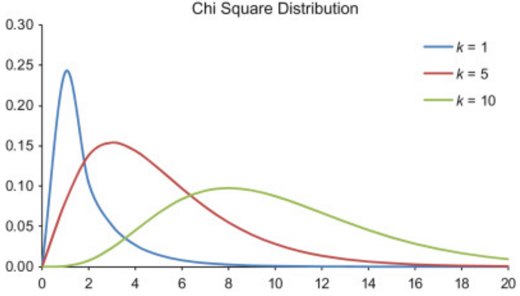
and compare with MGF of the gamma $\phi_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}, \qquad t < \lambda$

$$\Rightarrow \quad \chi_n^2 \sim \text{gamma}(\frac{n}{2}, \frac{1}{2}).$$

Additive property: the sum of two independent Chi square $\chi_{n_1}^2$ and $\chi_{n_2}^2$ is a Chi square $\chi_{n_1+n_2}^2$ with $n_1 + n_2$ degrees of freedom.

Chi square-distribution

Probability density function:



Note that:

- $Z \sim \mathrm{N}(0,1) \Rightarrow Z^2 \sim \chi^2_1$ (with one degree of freedom);
- $X_1, X_2, \ldots X_n$ are i.i.d. and $X_i \sim \mathrm{N}(\mu, \sigma^2)$,

$$\Rightarrow \sum_{i:1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2_n.$$

Example: We are attempting to locate a target in 3-dimensional space.

Suppose that the 3 coordinate errors (in meters) of the point chosen are independent random variables with mean 0 and variance 4.

Find the probability that the distance between the point chosen and the target exceeds 1 meter.

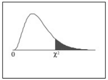
- X_i denotes the i-th coordinate error, $X_i \sim \mathrm{N}(0,4)$; $i : 1,2,3$ are independent.
- $D^2 = X_1^2 + X_2^2 + X_3^2$ represents the square of the distance of the point from the target.
- Note that $Z_i = \frac{X_i}{2} \sim \mathrm{N}(0,1)$; $i : 1,2,3$ and independent
  $\Rightarrow \frac{D^2}{4} = Z_1^2 + Z_2^2 + Z_3^2 = \chi_3^2$ is a Chi square with 3 degrees of freedom;

$$\mathbb{P}(\frac{D^2}{4}) > \frac{1}{4}) = \mathbb{P}(\chi_3^2 > \frac{1}{4}) =?$$

- How do we find the result?

Example: Chi square tables

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

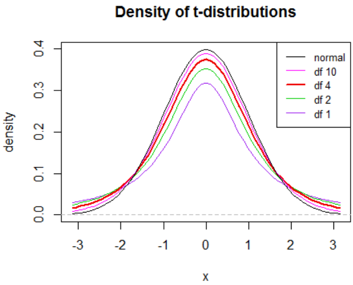
$$\mathbb{P}(\chi_3^2 > \frac{1}{4}) \cong \mathbb{P}(\chi_3^2 > 0.216) = 0.975$$

t-distribution

- If $Z \sim N(0,1)$ and χ^2_n are *independent*, the variable defined by

$$T_n = \frac{Z}{\sqrt{\frac{\chi^2_n}{n}}}$$

is said to have a *t-distribution* with *n degrees of freedom*

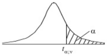


- it is *symmetric* (i.e. its density is symmetric) $\Rightarrow \mathbb{E}(T_n) = 0$.
- its density has *thicker tails* than the normal but as n becomes larger, it becomes more and more like the standard normal density.
- it can be shown that $\text{Var}(T_n) = \frac{n}{n-2}, \; n > 2$

Example: t-distribution tables

Table of the Student's *t*-distribution

The table gives the values of $t_{\alpha;\nu}$ where
$\Pr(T_\nu > t_{\alpha;\nu}) = \alpha$, with ν degrees of freedom



ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
1	3.078	6.314	12.076	31.821	63.657	318.310	636.620
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

Example: Find:

a) $\mathbb{P}(T_{12} \leq 1.356)$

b) $t_{0.025,9}$

c) $t_{0.975,9}$

$\Rightarrow$ a) $\mathbb{P}(T_{12} \leq 1.356) = 0.9$;

$\Rightarrow$ b) $t_{0.025,9} = 2.262$;

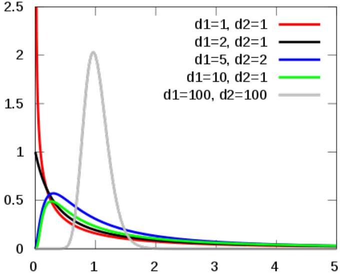
$\Rightarrow$ c) $t_{0.975,9} = -2.262$.

F-distribution

- If χ^2_n and χ^2_m are *independent*, the variable defined by

$$F_{n,m} = \frac{\chi^2_n/n}{\chi^2_m/m}$$

is said to have an *F-distribution* with *n* and *m degrees of freedom*



Example: F-distribution tables

Percentage points of *Fisher*'s distribution

$f_{0.05,v_1,v_2}$

v_2 \ v_1 (Degrees of freedom of the numerator (v_1))	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.64	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.31	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.55	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Degrees of freedom of the denominator (v_2)

Example: Find: a) $\mathbb{P}(F_{4,6} > 4.53)$ and $\mathbb{P}(F_{6,4} > \frac{1}{4.53})$; b) $f_{0.05,15,10}$.

a) $\mathbb{P}(F_{4,6} > 4.53) = 0.05$ and $\mathbb{P}(F_{6,4} > \frac{1}{4.53}) = 0.95$;

b) $f_{0.05,15,10} = 2.85$