STATISTICS

Module 2 *of*
MATHEMATICS AND STATISTICS

Marina Santacroce

marina.santacroce@unibocconi.it
Department of Decision Sciences, Bocconi University

L4. Probability spaces with equally likely outcomes. Combinatorics.

Outline

Sample Spaces with Equally Likely Outcomes

Today we will focus on examples of experiments with a **finite** number of **equally likely outcomes**.

The sample space $S = \{s_1, ...s_N\}$ and $\#(S) = N$ and the N outcomes s_i are *equally likely*. Thus,

- $\mathbb{P}(s_i) = \frac{1}{N} \quad i : 1, \ldots, N.$
- For any event $E \Rightarrow \quad \mathbb{P}(E) = \frac{\#(E)}{\#(S)}.$

Example:

A fair die is thrown twice and the number on each throw is recorded. What is the probability that the two numbers sum to seven?
and the probability of having twelve?

Example: communication system

A communication system consists of n (identical) lined up antennas.

The system is functional as long as *no* two consecutive antennas are *defective*.

If the defective antennas are m ($m \leq n$), what is the probability that *the system is functional*?

If $n = 4$ and $m = 2$

Denoting by 1 the defective antennas and by 0 the working ones.
There are 6 possible configurations:

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}$$

In 3 configurations (out of 6) the system is functional.
$\Rightarrow \mathbb{P}(\textit{system is functional}) = \frac{3}{6} = \frac{1}{2}$

but for general m and n? (How many are the possible configurations of antennas/the configurations with no two consecutive defective antennas?)

Basic Principle of Counting

Combinatorial analysis is the mathematical theory of *counting* the sets with specified characteristics.

Let us keep in mind the basic principle:

> If two experiments are to be performed. If experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, $\Rightarrow$ together there are mn possible outcomes of the two experiments.[1]

Examples:

- two consecutive tosses of a coin $\Longrightarrow$ the possible outcomes are 4
  - the first toss has 2 possible outcome $\{H, T\}$ and (for each of these) the second toss has two possible outcomes $\{H, T\}$
- how many different licence plates if the first 2 places are to be occupied by letters, the following 3 by numbers and the final 2 by letters? $\Longrightarrow 26^2 \cdot 10^3 \cdot 26^2$
- How many outcomes of the first two numbers playing Bingo? $\Longrightarrow 90 \cdot 89$

[1] the generalization to consider r experiments, each respectively with $n_1, n_2, \ldots n_r$ possible outcomes $\Rightarrow$ the r experiments have $n_1 n_2 \ldots n_r$ possible outcomes.

Permutations

Let $N = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct elements.

$n = \#(N)$ is the *cardinality* N.

Def. The set of all *permutations of N*, $\mathscr{P}(N)$, consists of all different **ordered arrangements of** N.

Example 1: Let $N = \{a, b, c\}$, we can enumerate the $\mathscr{P}(N)$:

$$\mathscr{P}(N) = \{abc, acb, bac, bca, cab, cba\}$$

How many permutaions of $N = \{a, b, c\}$? $\Rightarrow 6$
$\Rightarrow \#(\mathscr{P}(N)) = 3 \cdot 2 \cdot 1 = 3! = 6.$

$\Rightarrow$ There are $n \cdot (n-1) \cdot (n-2) \cdot \ldots 1 = n!$ different permutations of a set N of *n distinguishable* elements. Thus,

$$\boxed{\#(\mathscr{P}(N)) = n!}$$

Ex. 2 How many possible arrangements of 6 persons in a queue?

$\Rightarrow$ 6!

Permutations

If certain elements of N are indistinguishable from each other?

Example 1: How many letter arrangements of PEPPER?

- the permutations of $P_1 E_1 P_2 P_3 E_2 R$ are 6!
- since there are respectively 3! and 2! permutations of the P and of the E

$\Rightarrow \frac{6!}{3!2!} = 60$

In general, there are

$$\boxed{\#(\mathscr{P}(N)) = \frac{n!}{n_1!n_2!\dots n_r!}}$$

permutations of a set N with n elements of which n_1 are alike, n_2 are alike, ... and the last n_r are alike.

Ex. 2 How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags and 2 blue flags if all flags of the same color are identical?

$\Rightarrow \frac{9!}{4!3!2!} = 1260$

Ex. 3 How many arrangements of n antennas of which m defective?[2]

$\Rightarrow \frac{n!}{m!(n-m)!}$.

[2]Denote by 1 the defective antennas and by 0 the ones which are not

Combinations

Let $N = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct elements.

$n = \#(N)$ is the *cardinality* N.

Def. The set of *combinations of N of class k*, with $k \leq n$, $\mathscr{C}_{n,k}(N)$ is the set of different groups formed by k elements of N.[3]

Example 1: An urn contains 10 balls labeled from 1 to 10.
Two balls are drawn at a time. How many possible outcomes?

- $N = \{1, 2, \ldots 10\}$ e $\mathscr{C}_{10,2}(N) = \{(1,2), (1,3), \ldots (9,10)\}$,

$\Rightarrow \#(\mathscr{C}_{10,2}(N)) = \frac{10 \cdot 9}{2} = \frac{10!}{2!8!} = \binom{10}{2}$.

In general, there are

$$\boxed{\#(\mathscr{C}_{n,k}(N)) = \binom{n}{k}}$$

possible combinations of size k chosen from a set N of *n distinguishable elements*.

Ex. 2 A committee of 3 is to be formed from a group of 20 people. How many committees are possible? $\binom{20}{3}$.

[3] the order of the elements in the group is not relevant!

Example: communication system

A communication system is to consist of n (identical) lined up antennas.

The system is functional as long as *no* two consecutive antennas are *defective*.

If the defective antennas are m ($m \leq n$), what is the probability that *the system is functional*?

- Denote by "0" the $n - m$ no defective antennas

$$\wedge\ 0 \wedge\ 0 \wedge\ 0 \ldots \wedge\ 0 \wedge,$$

- any choice of m among the $n - m + 1$ $\wedge$ corresponds to a functional configuration.

$\Rightarrow$ Thus, they are $\binom{n-m+1}{m}$.

Thus,

$$\mathbb{P}(\textit{the system is functional}) = \frac{\binom{n-m+1}{m}}{\binom{n}{m}}$$

Remarks:

- there are $\binom{n}{k}$ permutations of the set $N = \{0, 0, \ldots, 1, 1\}$ with k “1” and $n - k$ “0”.
- $\binom{n}{k}$ is known as “binomial coefficient *n* choose *k*”[4].
- $\binom{n}{0} = \binom{n}{n} = 1$
- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- How many are the subsets of N (if we include also the empty set and N itself?
- How many ways to split n elements into two groups, respectively, of k and $n - k$ elements?

$\Rightarrow$ Let us generalize the last issue.

[4] $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Multinomial coefficients

A set of n distinct items is to be divided into r distinct groups of respective sizes n_1, n_2, ..., n_r, where $\sum_{i:1}^{r} n_i = n$.

How many different divisions are possible?

$$\Rightarrow \binom{n}{n_1} \cdot \binom{(n-n_1)}{n_2} \cdot \ldots \binom{(n_{r-1}+n_r)}{n_{r-1}} \cdot \binom{n_r}{n_r} = \frac{n!}{n_1! \cdot n_2! \ldots \ldots n_r!} = \binom{n}{n_1, \ldots, n_r}$$

$\binom{n}{n_1, \ldots, n_r}$ is known as *multinomial coefficient*.

Example: 9 children divide themselves into 3 teams of 3 each. How many divisions are possible if the teams are going to play in different leagues?

$$\Rightarrow \binom{9}{3,3,3}$$

Partitions of sets of indistinguishable items(*)

Let us divide a set N whose n elements are *indistinguishable* into r distinct groups.

How many divisions are possible?

Example: Seven (identical) candies are to be distributed among three children. How many distinct results are possible if each child is to receive at least one?

- Let us imagine to have the candies lined up and we want to divide them into 3 nonempty groups

$$0^\wedge 0^\wedge 0^\wedge 0^\wedge 0^\wedge 0^\wedge 0$$

- to do so, we can select 2 of the 6 spaces between the candies;
  i.e. any distinct division of the candies can be identified by a choice of 2 of the 6 $^\wedge$ as our division points $|$:
  choosing the first and the third $^\wedge$ $\Leftrightarrow$ $0|00|0000$ $\Leftrightarrow$ 1 candy to the first child, 2 to the second, 4 to the third.

$\Rightarrow$ There are $\binom{6}{2}$ possible choices of 2 numbers (positions of the $|$) among $\{1, 2, 3, 4, 5, 6\}$.

Partitions of sets of indistinguishable items(*)

Let us divide a set N whose n elements are *indistinguishable* into r distinct groups.

How many divisions are possible?

- Let us imagine to line up the n elements (denoted by 0) and and we want to divide them into r nonempty groups

$$0^\wedge 0^\wedge \ldots 0^\wedge 0^\wedge 0$$

- Similarly, any divisions into r groups corresponds to a choice of $r-1$ among the $n-1$ spaces $^\wedge$ between adjacent items.

$\Rightarrow$ Thus, the number of distinct divisions is $\binom{n-1}{r-1}$.

The number of integer solutions of equations

Recall the example, and denote by x_1, x_2, x_3 the number of candies given to the first, second and third child.

- Any possible division identifies an *integer positive solution* of the equation

$$x_1 + x_2 + x_3 = 7, \qquad x_i > 0.$$

Similarly, any *division of a set of n identical items* into r groups identifies an *integer positive solution* of

$$x_1 + x_2 + \ldots x_r = n, \tag{1}$$

$x_i > 0$, where x_i is the size of the i-th group.

Thus, the *number* of *integer positive solution of* (1) is $\binom{n-1}{r-1}$.

The number of nonnegative integer solutions of equations(*)

Example: How many divisions of 7 candies between 3 children, if no restrictions are imposed?

- Using the representation with “0” and “|”, any division can be represented by a permutation of the set of 7 “0” and 2 “|”:

  $|0|000000 \Leftrightarrow$ zero candies to the first, 1 to the second and 6 to the third.

$\Rightarrow$ Thus the possible divisions are $\binom{9}{2}$.

Similarly, any division of a set N of n indistinguishable items into *at most* r (distinct) groups can be represented by

- a *permutation* of n “0” and $r-1$ “|” either
- as a *non negative integer solution* of $x_1 + x_2 + ... + x_r = n$

  Since the number of permutations of the set of n “0” and $r-1$ “|” is $\binom{n+r-1}{r-1}$,

$\Rightarrow$ the *number* of *non negative integer solutions of equation* (1) is $\binom{n+r-1}{r-1}$.