Mathematics and Statistics (BEMACS) - A. Y. 2019/20

Lecture 1

BEMACS

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Set operations and properties

Let's start with a brief review of the terminology and notation about that will be used about sets. If an element x is in a set A, we write $x \in A$ (x is a member of A or x belongs to A). If x is not in A, we write $x \notin A$.

Definition 1.1. Let A, B be two sets. (a) A and B are said to be equal when they have the same elements; (b) Set A is a subset of B, indicated with $A \subseteq B$, if every element of A is also an element of B, that is

$$\forall x, \quad x \in A \Rightarrow x \in B$$

We also say that A is contained in B.

We write $A \subset B$ (A is a proper subset of B) when each element of A is also an element of B but at least one element of B that is not in A exists.

Property 1.2. Let A, B, C be sets. (a) $A \subseteq A$ (reflexive); (b) $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$ (antisymmetric); (c) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$ (transitivity).

A set is normally defined by either listing its elements explicitly or by specifying a property (or more than one) that determines the elements of the set.

Set operations and properties

Definition 1.3. Let A, B be sets. The union and the intersection of A and B, indicated by $A \cup B$ and $A \cap B$, are sets defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \qquad A \cap B = \{x : x \in A \text{ and } x \in B\}$$

The difference of B relative to A, indicated by $A \setminus B$, is defined as follows

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

The set with no elements is called the empty set and is denoted by the symbol $\emptyset$. A and B are said to be disjoint if they have no elements in common, i.e. $A \cap B = \emptyset$.

Remark 1.4. From the definition 1.3. we have:

- the difference of A relative to B, indicated by $B \setminus A$, is defined as follows

$$B \setminus A = \{x : x \in B \text{ and } x \notin A\}$$

- the union of A and B can be written as follows

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$$

  that is as union of disjoint sets.

Set operations and properties

Two useful properties that combine the operations of union and intersection are given next.

Property 1.5. Let A, B, C be sets. We have:

- $\cap$ distributes over $\cup$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- $\cup$ distributes over $\cap$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

From now on we consider sets A, B as subsets of X (which is called universe set). Therefore we can define the operation which is called the complement of A with respect to X, that is

$$A^c = X \setminus A = \{x \in X : x \notin A\}$$

Set operations and properties

Remark 1.6. Let A, $B \subseteq X$ be sets. We have the following properties:

(a) $\emptyset^c = X$, $X^c = \emptyset$ (b) $(A^c)^c = A$ (c) $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$

(d) $A \cup A^c = X$ (e) $A \cap A^c = \emptyset$ (f) $A \cup X = X$

(g) $A \cap X = A$ (h) $A \cup B = B \cup A$ (i) $A \cap B = B \cap A$

Property 1.7. Let A, $B \subseteq X$ be sets. We have

(a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$

which are called the De Morgan's laws.
Proof. (a) We must show that $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$.
Let x be any element of $(A \cup B)^c$, we have

$$x \notin A \cup B \Rightarrow x \notin A \text{ or } x \notin B \Rightarrow x \in A^c \text{ and } x \in B^c \Rightarrow A^c \cap B^c$$

Let x be any element of $A^c \cap B^c$, we have

$$x \in A^c \text{ and } x \in B^c \Rightarrow x \notin A \text{ or } x \notin B \Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)^c$$

Sets of numbers and intervals

Several special sets are used and they are indicated by standard notations (we will use the symbol $:=$ to mean that the symbol on the left is being defined by the symbol on the right). We have:

- Sets of natural numbers $\mathbb{N}$ and integers $\mathbb{Z}$

$$\mathbb{N} := \{0, 1, 2, 3, ...\} \quad \text{and} \quad \mathbb{Z} := \{0, 1, -1, 2, -2, 3, ...\}$$

- The set of rational numbers $\mathbb{Q} := \{m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$
- The set of real numbers $\mathbb{R}$ (this set is the union of the set of rational numbers and the set of irrational numbers).

We know that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

The are special subsets of $\mathbb{R}$ which are called intervals. Let $a, b \in \mathbb{R}$ be real numbers, we have:

- $[a, a] = \{a\}$ (singleton a), $(a, a) = \emptyset$ (empty set);
- $[a, b]$ (closed and bounded interval); (a, b) (open and bounded interval); $[a, b)$ and $(a, b]$ (bounded intervals);
- $[a, +\infty)$, $(a, +\infty)$, $(-\infty, b]$, $(-\infty, b)$ and $(-\infty, +\infty)$ (unbounded intervals).

Convex set

Example 1.8. Let's consider sets $A = (-\infty, 10]$ and $B = (-20, +\infty)$, where $\mathbb{R} = (-\infty, +\infty)$ is the universal set. We have

$$A^c = \mathbb{R} \setminus A = (10, +\infty) \qquad B^c = \mathbb{R} \setminus B = (-\infty, -20] \qquad A^c \cap B^c = \emptyset$$

Suppose two points x_1, x_2 of $\mathbb{R}$ such that $x_1 \neq x_2$. Points of the form

$$x = \alpha x_1 + (1 - \alpha)\, x_2, \qquad \alpha \in \mathbb{R}$$

form the line passing through x_1 and x_2. The parameter value $\alpha = 0$ corresponds to $x = x_2$, and $\alpha = 1$ corresponds to $x = x_1$. Values of α between 0 and 1 correspond to the closed line segment between x_1 and x_2.

Definition 1.9. A set A is convex if the line segment between any two points in A lies in A, that is, if for any x_1, $x_2 \in A$ and any α with $\alpha \in [0, 1]$ we have

$$\alpha x_1 + (1 - \alpha)\, x_2 \in A$$

We call a point of the form $\alpha_1 x_1 + \alpha_2 x_2$, where $\alpha_1 + \alpha_2 = 1$ and $\alpha_1, \alpha_2 \geq 0$, a convex linear combination of the points x_1, x_2.

Every interval (closed or open, bounded or unbounded) is a convex set.

The extended real line

The set of real numbers $\mathbb{R}$ is an ordered set, it means that for any a, b real numbers we can always say when $a < b$ (a is less than b), $a = b$ (a, b are equal) or $a > b$ (a is greater than b). It is convenient to extend the real numbers by adjoining two new points $+\infty$ and $-\infty$. The resulting set, indicated with $\overline{\mathbb{R}}$, is called the extended real number set

$$\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\} \qquad \text{or} \qquad \overline{\mathbb{R}} := [-\infty, +\infty]$$

where $+\infty$ is greater than every other real number and $-\infty$ is less than every real number. This makes the extended real number set an ordered set. We can introduce some calculation conventions with $+\infty$ and $-\infty$ symbols (x is a real number) as follows

$$+\infty + (+\infty) = +\infty \qquad -\infty + (-\infty) = -\infty \qquad x + (\pm\infty) = \pm\infty$$

$$(\pm\infty) \cdot (\pm\infty) = +\infty \qquad (\pm\infty) \cdot (\mp\infty) = -\infty \qquad x \cdot \infty = \infty \ (x \neq 0)$$

$$x/\infty = 0 \qquad \infty/x = \infty \qquad x/0 = \infty \ (x \neq 0)$$

that is we have a partial arithmetization of the symbols $+\infty$ and $-\infty$.

Absolute value and distance

Definition 1.10. Let x be a real number. The non-negative real number, indicated by $|x|$, defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

is called the absolute value (or modulus) of x.

The absolute value of x has a geometric interpretation: it represents the distance of point x from the origin of the real line.

Property 1.11. Let x, y be real numbers. We have

- $|x| \geq 0$, $|x| = 0 \Leftrightarrow x = 0$;
- $\sqrt{x^2} = |x|$, $x^2 = |x|^2$, $|x| = \max(x, -x)$
- $|xy| = |x|\,|y|$, $\left|xy^{-1}\right| = |x|\left|y^{-1}\right|$ with $y \neq 0$
- $|x + y| \leq |x| + |y|$ (triangle inequality).

Remark 1.12. $k \geq 0$. (a) $|x| = k$ if and only if $x = -k$ or $x = k$, opposite points with respect to the origin have same distance; (b) $|x| \leq k$ if and only if $-k \leq x \leq k$; (c) $|x| \leq k$ if and only if $x \leq -k$ or $x \geq k$.

Examples

Example 1.13. Let x be a real number. Show that $|x| = \max(x, -x)$.
We have to discuss two cases:

- if $x \geq 0$ then $|x| = x$ and $\max(x, -x) = x$;
- if $x < 0$ then $|x| = -x$ and $\max(x, -x) = -x$;

therefore we conclude that for any $x \in \mathbb{R}$, $|x| = \max(x, -x)$.

Example 1.14. Let x be a real number. Solve $|x - 1| + |2 - 2x| \geq 4$.
Using the absolute value properties, we get

$$|x-1| + 2|x-1| \geq 4 \quad \Rightarrow \quad |x-1| \geq \frac{4}{3} \quad \Rightarrow \quad x - 1 \leq -\frac{4}{3} \text{ or } x - 1 \geq \frac{4}{3}$$

from which the solution set is $x \leq -1/3$ or $x \geq 7/3$.

Example 1.15. Let x be a real number. Solve $|2x| + |-3x| \leq 10$.
We have

$$|2x| + |-3x| \leq 10 \quad \Rightarrow \quad 5|x| \leq 10 \quad \Rightarrow \quad |x| \leq 2 \quad \Rightarrow \quad -2 \leq x \leq 2$$

Distance between two points

Definition 1.16. Let x, y be real numbers. The non-negative real number, indicated by $d\left(x, y\right)$, defined by

$$d(x,y) = \begin{cases} y - x & y \geq x \\ -\left(y - x\right) & y < x \end{cases}$$

is called the distance between points x and y.

The distance between points x and y is equal to the absolute value between x and y, that is

$$d(x,y) = |y - x|$$

and $d(0,x) = |x|$.

Property 1.17. Let x, y, z be real numbers. We have

(a) $\forall x, y,\ d(x,y) \geq 0,\ d(x,y) = 0 \Leftrightarrow x = y$ (non-negativity);
(b) $\forall x, y,\ d(x,y) = d(y,x)$ (symmetry);
(c) $\forall x, y, z,\ d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality).

Proof. (b) We have

$$d(x,y) = |y - x| = |x - y| = d(y,x)$$

Neighbourhood of a point

Definition 1.18. Let x_0 and ε be real numbers with $\varepsilon > 0$. The set of points, indicated by $B_\varepsilon(x_0)$, defined by

$$B_\varepsilon(x_0) = \{x \in \mathbb{R} : \mathrm{d}\,(x, x_0) < \varepsilon\}$$

is called the neighbourhood of x_0 with radius ε.

From the definition 1.18 we can observe that

$$\begin{aligned} \mathrm{d}\,(x, x_0) < \varepsilon \quad &\Rightarrow \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad -\varepsilon < x - x_0 < \varepsilon \\ &\Rightarrow \quad x_0 - \varepsilon < x < x_0 + \varepsilon \end{aligned}$$

therefore $B_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon)$ (i.e. $B_\varepsilon(x_0)$ is an open interval).

Remark 1.19. (a) If $x_0 = 0$ then $B_\varepsilon(0) = (-\varepsilon, \varepsilon)$ (i.e. a neighbourhood of 0 with radius $\varepsilon > 0$); (b) any $x_0 \in \mathbb{R}$ has infinitely many neighbourhoods $B_\varepsilon(x_0)$, one for each value of $\varepsilon > 0$. (c) every open interval $(a, +\infty)$ is called a neighbourhood of $+\infty$ and every open interval $(-\infty, a)$ is called a neighbourhood of $-\infty$.

Examples

Example 1.20. The neighbourhood of $x_0 = 4$ and radius $\varepsilon = 1$ is

$$B_1(4) = \{x \in \mathbb{R} : \mathrm{d}(x, 4) < 1\}$$

We have

$$\mathrm{d}(x, 4) < 1 \quad \Rightarrow \quad |x - 4| < 1 \quad \Rightarrow \quad 4 - 1 < x < 4 + 1$$

therefore $B_1(4) = (3, 5)$. For example, $7/2 \in B_1(4)$ and $11/2 \notin B_1(4)$.

Example 1.21. The neighbourhood of $x_0 = 0$ and radius $\varepsilon = 1$ is

$$B_1(0) = \{x \in \mathbb{R} : \mathrm{d}(x, 0) < 1\} = (0 - 1, 0 + 1) = (-1, 1)$$

We have

$$\mathrm{d}(x, 0) < 1 \quad \Rightarrow \quad |x| < 1 \quad \Rightarrow \quad -1 < x < 1$$

therefore $B_1(0) = (-1, 1)$. For example, $1/2 \in B_1(0)$ and $-3/2 \notin B_1(0)$.