Mathematics and Statistics (BEMACS) - A. Y. 2019/20

Lecture 2

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Vectors

Definition 2.1. Let $x_1, x_2, ..., x_n$ be real numbers. A vertical order or an horizontal order of $x_1, x_2, ..., x_n$, indicated with $\mathbf{x}$, that is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{or} \qquad \mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

is called a column vector or a row vector. Numbers $x_1, x_2, ..., x_n$ are said to be the components of the column/row vector. The number of components n is said to be the dimension of the vector. The set of all the vectors with n components is indicated with $\mathbb{R}^n$ (n-dimensional real coordinate space).

Example 2.2. Consider the following vectors

$$\mathbf{x}^1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^2 = \begin{bmatrix} -1 & 3 & 0 & 1 \end{bmatrix}$$

where $\mathbf{x}^1 \in \mathbb{R}^3$ and $\mathbf{x}^2 \in \mathbb{R}^4$.

Operations with vectors and properties

Definition 2.3. Let $\mathbf{x} \in \mathbb{R}^n$ be a column/row vector. The row/column vector, indicated with $\mathbf{x}^{\mathrm{T}}$, that is

$$\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \qquad \text{or} \qquad \mathbf{x}^{\mathrm{T}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is called the transpose vector of $\mathbf{x}$.

Remark 2.4. (a) $\left(\mathbf{x}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{x}$; (b) We indicate with $\mathbf{0}$ the null vector, that is the vector with null components; (c) We indicate with $\mathbf{e}^i$, with $i = 1, 2, ..., n$, the vector which has 1 in position i and 0 in the others (standard unit vectors or versors of $\mathbb{R}^n$), that is

$$\mathbf{e}^1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^{\mathrm{T}}, \quad \cdots \quad , \quad \mathbf{e}^n = \begin{pmatrix} 0 & 0 & \cdots & 1 \end{pmatrix}^{\mathrm{T}}$$

(d) Two vectors $\mathbf{x}^1$, $\mathbf{x}^2$, with the same dimension n, are equal if and only if $x_i^1 = x_i^2$ with $i = 1, 2, ..., n$.

Operations with vectors and properties

Definition 2.5. Let $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$ be vectors. The sum of $\mathbf{x}^1$ and $\mathbf{x}^2$, indicated with $\mathbf{x}^1 + \mathbf{x}^2$, is the vector on $\mathbb{R}^n$ defined by

$$\mathbf{x}^1 + \mathbf{x}^2 = \begin{bmatrix} x_1^1 + x_1^2 \\ x_2^1 + x_2^2 \\ \vdots \\ x_n^1 + x_n^2 \end{bmatrix}$$

where each component is the sum of components $\mathbf{x}^1$ and $\mathbf{x}^2$ with the same position.

Property 2.6. Let $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3 \in \mathbb{R}^n$ be vectors. We have:

1. $\mathbf{x}^1 + \mathbf{x}^2 = \mathbf{x}^2 + \mathbf{x}^1$;
2. $\left(\mathbf{x}^1 + \mathbf{x}^2\right) + \mathbf{x}^3 = \mathbf{x}^1 + \left(\mathbf{x}^2 + \mathbf{x}^3\right)$;
3. $\mathbf{x}^1 + \mathbf{0} = \mathbf{0} + \mathbf{x}^1$ ($\mathbf{0}$ is the neutral element);
4. $\mathbf{x}^1 + \left(-\mathbf{x}^1\right) = \left(-\mathbf{x}^1\right) + \mathbf{x}^1 = \mathbf{0}$ ($-\mathbf{x}$ is called the opposite of $\mathbf{x}$).

Proof. (a) For any $i = 1, 2, ..., n$, We have

$$\mathbf{x}^1 + \mathbf{x}^2 = \left[x_i^1 + x_i^2\right] = \left[x_i^2 + x_i^1\right] = \left[x_i^2\right] + \left[x_i^1\right] = \mathbf{x}^2 + \mathbf{x}^1$$

Operations with vectors and properties

Definition 2.7. Let $\mathbf{x} \in \mathbb{R}^n$ be a vector and α a real number (or scalar). The product $\alpha\mathbf{x}$ is the vector on $\mathbb{R}^n$ given by

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

where each component of $\mathbf{x}$ is multiplied by α.

Property 2.8. Let $\mathbf{x}^1$, $\mathbf{x}^2 \in \mathbb{R}^n$ be vectors. For any α, β real numbers, we have:

1. $\alpha \left(\mathbf{x}^1 + \mathbf{x}^2\right) = \alpha\mathbf{x}^1 + \alpha\mathbf{x}^2$;
2. $(\alpha + \beta)\,\mathbf{x}^1 = \alpha\mathbf{x}^1 + \beta\mathbf{x}^1$;
3. $1 \cdot \mathbf{x}^1 = \mathbf{x}^1$ (1 is the neutral element);
4. $\alpha\left(\beta\mathbf{x}^1\right) = (\alpha\beta)\,\mathbf{x}^1$.

Proof. (a) For any $i = 1, 2, ..., n$, We have

$$\alpha\left(\mathbf{x}^1 + \mathbf{x}^2\right) = \alpha\left[x_i^1 + x_i^2\right] = \left[\alpha x_i^1 + \alpha x_i^2\right] = \left[\alpha x_i^1\right] + \left[\alpha x_i^2\right] = \alpha\mathbf{x}^1 + \alpha\mathbf{x}^2$$

Examples

Example 2.9. Consider on $\mathbb{R}^3$ the following vectors

$$\mathbf{x}^1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{x}^2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \qquad \mathbf{x}^3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The sum $\mathbf{x}^1 + \mathbf{x}^2$ and the product $-2\mathbf{x}^3$ are given by

$$\mathbf{x}^1 + \mathbf{x}^2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \qquad -2 \cdot \mathbf{x}^3 = (-2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$$

The sum $\mathbf{x}^1 + \mathbf{x}^2 + \mathbf{x}^3$ on $\mathbb{R}^3$ is equal to

$$\mathbf{x}^1 + \mathbf{x}^2 + \mathbf{x}^3 = \left(\mathbf{x}^1 + \mathbf{x}^2\right) + \mathbf{x}^3 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Inner product of vectors

Definition 2.10. Let $\mathbf{x}^1$, $\mathbf{x}^2 \in \mathbb{R}^n$ be vectors. Tha real number (or scalar), defined by $\left(x^1, x^2\right)$, given by

$$\left(\mathbf{x}^1, \mathbf{x}^2\right) := \sum_{s=1}^{n} x_s^1 \, x_s^2 = x_1^1 \cdot x_1^2 + \cdots + x_n^1 \cdot x_n^2$$

is said to be the inner product (or the scalar product) of $\mathbf{x}^1$ and $\mathbf{x}^2$.

Property 2.11. Let $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3 \in \mathbb{R}^n$ be vectors. We have:

(a) $\left(\mathbf{x}^1, \mathbf{x}^2\right) = \left(\mathbf{x}^2, \mathbf{x}^1\right)$ (symmetry)

(b) $\left(\mathbf{x}^1 + \mathbf{x}^2, \mathbf{x}^3\right) = \left(\mathbf{x}^1, \mathbf{x}^3\right) + \left(\mathbf{x}^2, \mathbf{x}^3\right)$ (additivity)

(c) $\left(\alpha \mathbf{x}^1, \beta \mathbf{x}^2\right) = \alpha\beta \left(\mathbf{x}^1, \mathbf{x}^2\right)$, for any $\alpha, \beta \in \mathbb{R}$ (homogeneity).

Proof. (a) From the definitio of inner product, we have

$$\left(\mathbf{x}^1, \mathbf{x}^2\right) = \sum_{s=1}^{n} x_s^1 \, x_s^2 = \sum_{s=1}^{n} x_s^2 \, x_s^1 = \left(\mathbf{x}^2, \mathbf{x}^1\right)$$

Examples

Remark 2.12. If either $\mathbf{x}^1$ or $\mathbf{x}^2$ is th $\mathbf{0}$ vector, then $\left(\mathbf{x}^1, \mathbf{x}^2\right) = \mathbf{0}$. The converse is false.

Example 2.13. Consider vectors $\mathbf{x}^1 = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and $\mathbf{x}^2 = \begin{pmatrix} -1 & -1 \end{pmatrix}$. We have

$$\left(\mathbf{x}^1, \mathbf{x}^2\right) = 1 \cdot (-1) + (-1) \cdot (-1) = -1 + 1 = 0$$

but $\mathbf{x}^1$ and $\mathbf{x}^2$ are not equal to $\mathbf{0}$.

Example 2.14. Consider $\mathbf{x}^1 = \mathbf{0}$ and $\mathbf{x}^2 \in \mathbb{R}^n$. For any $\mathbf{x}^2 \in \mathbb{R}^n$, the inner product of $\mathbf{x}^1$ and $\mathbf{x}^2$ is

$$\left(\mathbf{x}^1, \mathbf{x}^2\right) = \left(\mathbf{0}, \mathbf{x}^2\right) = 0$$

therefore the null vector is orthogonal to any vector of $\mathbb{R}^n$.

Orthogonal vectors

Definition 2.15. Let $\mathbf{x}^1$, $\mathbf{x}^2 \in \mathbb{R}^n$ be vectors. Vectors $x1$ and $x2$ such that

$$\left(\mathbf{x}^1, \mathbf{x}^2\right) := \sum_{s=1}^{n} x_s^1 \, x_s^2 = 0$$

are called orthogonal vectors.

Example 2.16. Let α be a real number (a parameter). Consider the following vectors

$$\mathbf{x}^1 = \begin{bmatrix} \alpha \\ 2 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^2 = \begin{bmatrix} \alpha \\ -\alpha \\ 3 \end{bmatrix}$$

The inner product between $\mathbf{x}^1$ and $\mathbf{x}^2$ is equal to

$$\left(\mathbf{x}^1, \mathbf{x}^2\right) = \sum_{s=1}^{3} x_s^1 \, x_s^2 = \alpha^2 - 2\alpha - 3$$

which depends on the value of α. Vectors $\mathbf{x}^1$ and $\mathbf{x}^2$ are orthogonal if and only if $\left(\mathbf{x}^1, \mathbf{x}^2\right) = 0$, that is

$$\left(\mathbf{x}^1, \mathbf{x}^2\right) = 0 \Leftrightarrow \alpha^2 - 2\alpha - 3 = 0 \Leftrightarrow \alpha = -1 \text{ or } \alpha = 3$$

Linear combination of vectors

Definition 2.17. Let $\mathbf{x}^1$, $\mathbf{x}^2$, ..., $\mathbf{x}^n \in \mathbb{R}^n$ be vectors. The vector $\mathbf{x}$ given by

$$\mathbf{x} = \sum_{s=1}^{n} \alpha_s \mathbf{x}^s = \alpha_1 \mathbf{x}^1 + \alpha_2 \mathbf{x}^2 + \cdots + \alpha_n \mathbf{x}^n$$

is called the linear combination of $\mathbf{x}^1$, $\mathbf{x}^2$, ..., $\mathbf{x}^n$ with coefficients $\alpha_1, \alpha_2, ..., \alpha_n$.

Example 2.18. Consider on $\mathbb{R}^3$ the following vectors

$$\mathbf{x}^1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{x}^2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \qquad \mathbf{x}^3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The vector $2x^1 - 3x^2 + 2x^3$ on $\mathbb{R}^3$, i. e. the linear combination of $\mathbf{x}^1$, x^2 and x^3 with coefficients, respectively, 2, -3 and 2, is given by

$$2\mathbf{x}^1 - 3\mathbf{x}^2 + 2\mathbf{x}^3 = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 0 \end{bmatrix}$$

Convex linear combinations and convex sets

Definition 2.19. Let $\mathbf{x}^1$, $\mathbf{x}^2$, ..., $\mathbf{x}^n \in \mathbb{R}^n$ be vectors. The vector $\mathbf{x}$ given by

$$\mathbf{x} = \sum_{s=1}^{n} \alpha_s \mathbf{x}^s = \alpha_1 \mathbf{x}^1 + \alpha_2 \mathbf{x}^2 + \cdots + \alpha_n \mathbf{x}^n$$

is called the convex linear combination of $\mathbf{x}^1$, $\mathbf{x}^2$, ..., $\mathbf{x}^n$ with coefficients $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ and $\alpha_1 + \cdots + \alpha_n = 1$.

Example 2.20. Consider on $\mathbb{R}^3$ the following vectors

$$\mathbf{x}^1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{x}^2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \qquad \mathbf{x}^3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The vector $1/4\mathbf{x}^1 + 1/4\mathbf{x}^2 + 1/2\mathbf{x}^3$ on $\mathbb{R}^3$, i. e. the convex linear combination of $\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$ with coefficients, respectively, $1/4$, $1/4$ and $1/2$, is given by

$$\frac{1}{4}\mathbf{x}^1 + \frac{1}{4}\mathbf{x}^2 + \frac{1}{2}\mathbf{x}^3 = \frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 0 \\ 1/4 \end{bmatrix}$$

Norm of a vector

Definition 2.21. Let $\mathbf{x}^1 \in \mathbb{R}^n$ be vector. The non-negative real number (or non-negative scalar), indicated with $\|\mathbf{x}\|$, defined by

$$\|\mathbf{x}\| = \sqrt{\sum_{s=1}^{n} \left(x_s^1\right)^2} = \sqrt{\left(x_1^1\right)^2 + \left(x_2^1\right)^2 + \cdots + \left(x_n^1\right)^2}$$

is said to be the norm of $\mathbf{x}^1$.

Remark 2.22. (a) The norm of $\mathbf{x}$ represents the measure of the distance between $\mathbf{x}$ and $\mathbf{0}$ (the origin of $\mathbb{R}^n$); (b) If $n = 1$ the norm of $\mathbf{x} \in \mathbb{R}$ is equal to the absolute value of x; (c) the norm fo $\mathbf{x}^1$ can be defined using the definition of inner product, in fact we have

$$\|\mathbf{x}\| = \sqrt{\sum_{s=1}^{n} \left(x_s^1\right)^2} = \sqrt{\sum_{s=1}^{n} \left(x_s^1 \cdot x_s^1\right)} = \sqrt{\left(\mathbf{x}^1, \mathbf{x}^1\right)} = \left(\mathbf{x}^1, \mathbf{x}^1\right)^{1/2}$$

Example 2.23. Consider $\mathbf{x}^1 = \begin{pmatrix} -1 & 2 & 1 & -3 \end{pmatrix} \in \mathbb{R}^4$. We have

$$\|\mathbf{x}\| = \sqrt{\sum_{s=1}^{n} \left(x_s^1\right)^2} = \sqrt{\left(-1\right)^2 + 2^2 + 1^2 + \left(-3\right)^2} = \sqrt{13}$$

Properties of a norm of a vector

Property 2.24. Let $\mathbf{x}^1$, $\mathbf{x}^2 \in \mathbb{R}^n$ be vectors and α a real number. We have:

(a) $\left\|\mathbf{x}^1\right\| \geq 0$, $\left\|\mathbf{x}^2\right\| = 0$ if and only if $x = 0$ (non-negativity);

(b) $\left\|\alpha\mathbf{x}^1\right\| = |\alpha| \left\|\mathbf{x}^1\right\|$ (absolute homogeneity);

(c) $\left\|\mathbf{x}^1+\mathbf{x}^2\right\| \leq \left\|\mathbf{x}^1\right\| + \left\|\mathbf{x}^2\right\|$ (subadditivity).

Proof. (b) From the definition of norm, we have

$$\begin{aligned} \left\|\alpha\mathbf{x}^1\right\| &= \sqrt{\sum_{s=1}^n \left(\alpha x_s^1\right)^2} = \sqrt{\sum_{s=1}^n \left(\alpha x_s^1 \cdot \alpha x_s^1\right)} = \sqrt{\alpha^2 \sum_{s=1}^n \left(x_s^1 \cdot x_s^1\right)} \\ &= |\alpha| \sqrt{\sum_{s=1}^n \left(x_s^1 \cdot x_s^1\right)} = |\alpha| \left(\mathbf{x}^1, \mathbf{x}^1\right)^{1/2} = |\alpha| \, \|\mathbf{x}\| \end{aligned}$$

Remark 2.25. We can generalize property 2.24 (c) as follows

$$\left\|\mathbf{x}^1+\mathbf{x}^2 + \cdots + \mathbf{x}^n\right\| \leq \left\|\mathbf{x}^1\right\| + \left\|\mathbf{x}^2\right\| + \cdots \|\mathbf{x}^n\|$$

for any $\mathbf{x}^1$, $\mathbf{x}^2$, ..., $\mathbf{x}^n \in \mathbb{R}^n$.