Mathematics and Statistics (BEMACS) - A. Y. 2019/20

Lecture 4

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Interior, exterior and boundary points

From the definition of $B_\varepsilon(\mathbf{x}^0)$ (i.e. the neighbourhood of $\mathbf{x}^0 \in \mathbb{R}^n$ with radius $\varepsilon > 0$), it is possible to introduce a small classification about points of any subset of $\mathbb{R}^n$ ($A \subseteq \mathbb{R}^n$).

Definition 4.1. Let $A \subseteq \mathbb{R}^n$. A point $\mathbf{x}^0 \in A$ is said to be an interior point of A if $\varepsilon > 0$ exists such that $B_\varepsilon(\mathbf{x}^0) \subseteq A$ (i.e. $B_\varepsilon(\mathbf{x}^0)$ is entirely contained in A).

Example 4.2. Let $A = (2, 5)$. Is $x^0 = 3$ an interior point of A? Yes, because at least a $B_\varepsilon(3)$, with $\varepsilon > 0$, exists such that $B_\varepsilon(3) \subseteq A$. For example, $B_{1/2}(3) \subseteq A$.

Definition 4.3. Let $A \subseteq \mathbb{R}^n$. A point $\mathbf{x}^0 \in \mathbb{R}^n$ is said to be an exterior point of A if $\varepsilon > 0$ exists such that $B_\varepsilon(\mathbf{x}^0) \subseteq A^c$ (i.e. $B_\varepsilon(\mathbf{x}^0)$ is entirely contained in A^c).

Example 4.4. Let $A = (2, 5)$. Is $x^0 = 7$ an exterior point of A? Yes, because at least a $B_\varepsilon(7)$, with $\varepsilon > 0$, exists such that $B_\varepsilon(7) \subseteq A^c$. For example, $B_{1/2}(7) \subseteq A^c$.

Form now on, we indicate with $\text{int}(A)$ (read as interior of A) the set of all the interior points of A. Therefore $\text{int}(A^c)$ will be the set of all the interior (exterior) points of A^c (A). We have

$$int(A) \subseteq A \qquad int(A^c) \subseteq A^c \qquad int(A^c)^c = int(A)$$

Interior, exterior and boundary points

Example 4.5. Let $A = (3, +\infty)$. Set A is only made up of interior points, therefore we have

$$\text{int}(A) = (3, +\infty) = A \qquad \text{and} \qquad \text{int}(A^c) = (-\infty, 3) \subseteq A^c$$

because it is $A^c = (-\infty, 3]$ and point 3 is not an interior point of A^c (it is on the border of A^c).

Definition 4.6. Let $A \subseteq \mathbb{R}^n$. A point $\mathbf{x}^0 \in \mathbb{R}^n$ is said to be a boundary point of A if

$$\forall \varepsilon > 0, \quad B_\varepsilon(\mathbf{x}^0) \cap A \neq \emptyset \qquad \text{and} \qquad B_\varepsilon(\mathbf{x}^0) \cap A^c \neq \emptyset$$

that is $\mathbf{x}^0$ is neither an interior point nor an exterior point of A.

Remark 4.7. $\mathbf{x}^0 \in \mathbb{R}^n$ is a boundary point of $A \subseteq \mathbb{R}^n$ if all its neighbourhoods contain both points of A and points of A^c. From now on, we will indicate with ∂A (read as boundary of A) the set of all the boundary points of A.

Example 4.8. Let $A = \left\{\mathbf{x} \in \mathbb{R}^2 : x_2 \geq x_1^2\right\}$. We have

$$\text{int}(A) = \left\{\mathbf{x} \in \mathbb{R}^2 : x_2 > x_1^2\right\} \quad \partial A = \left\{\mathbf{x} \in \mathbb{R}^2 : x_2 = x_1^2\right\}$$

Interior, exterior and boundary points

Remark 4.9. (a) For any set $A \subseteq \mathbb{R}^n$, each point of A is either an interior point, or exterior point or a boundary point. **(b)** Let $A \subseteq \mathbb{R}$ be a bounded set. Then

$$\sup A \in \partial A \qquad \text{and} \qquad \inf A \in \partial A$$

Example 4.10. Let $A = \left\{\mathbf{x} \in \mathbb{R}^2 : |x| < 4\right\}$ and $B = \left\{\mathbf{x} \in \mathbb{R}^2 : |x| \geq 2\right\}$. In $\mathbb{R}$, for A (bounded) we have

$$int(A) = (-4, 4) \quad \partial A = \{-4, 4\} \quad \sup A = 4 \in \partial A \quad \inf A = -4 \in \partial A$$

and for B (unbounded) we have

$$int(B) = (-\infty, 2] \cup [2, +\infty) \quad \partial B = \{-2, 2\} \quad \nexists \sup B \quad \nexists \inf B$$

Consider $A \cap B$, we have

$$int(A \cap B) = (-4, -2) \cup (2, 4) \quad \partial\left(A \cap B\right) = \{-4, -2, 2, 4\}$$

and $\inf\left(A \cap B\right) = -4$ and $\sup\left(A \cap B\right) = 4$.

Isolated points

Among all the boundary points there is a special subset which is called the set of isolated points.

Definition 4.11. Let $A \subseteq \mathbb{R}^n$. A point $\mathbf{x}^0 \in A$ is said to be an isolated point of A if

$$\exists B_\varepsilon(\mathbf{x}^0) \text{ such that } B_\varepsilon(\mathbf{x}^0) \cap A = \left\{\mathbf{x}^0\right\}$$

that is if at least a neighbourhood of $\mathbf{x}^0$ exists which does not have other points of A except $\mathbf{x}^0$.

Remark 4.12. Isolated points are boundary points (the opposite is not true).

Example 4.13. Let $A = (-2, 10] \cup \{12\}$ be a bounded subset of $\mathbb{R}$. Point $x^0 = 12 \in A$ (an element of A) and

$$\exists B_\varepsilon(12) \text{ such that } B_\varepsilon(12) \cap A = \{12\}$$

therefore $x^0 = 12$ is an isolated point of A. From $\partial A = \{-2, 10, 12\}$ we observe that $x^0 = 12 \in \partial A$, while $x^1 = -2 \in \partial A$ but is not an isolated point of A. We have

$$\sup A = 12 \in \partial A \quad \inf A = -2 \in \partial A$$

with $\max(A) = 12$ and $\nexists \min(A)$.

Limit (or cluster, accumulation) points

Definition 4.14. Let $A \subseteq \mathbb{R}^n$. A point $\mathbf{x}^0 \in \mathbb{R}^n$ is said to be a limit point of A if each neighbourhood $B_\varepsilon(\mathbf{x}^0)$ has at least one point of A different from $\mathbf{x}^0$.

Set A' will indicate the set of all limit points of A. It is said to be the derived set of A.

Property 4.15. Let $A \subseteq \mathbb{R}^n$. We have:

1. any interior point of A is a limit point of A;
2. any boundary point of A is a limit point of A if and only if it is not isolated.

Theorem 4.16. Each neighbourhood of a limit point of A has infinitely many points of A.

Example 4.17. Let $A = (-12, 10] \cup \{12\}$ be a bounded subset of $\mathbb{R}$. The set of all the limit points of A is

$$\operatorname{int}(A) = (-12, 10) \quad \partial A = \{-12, 10, 12\} \quad A' = [-12, 10]$$

where -12 does not belong to A but it is a limit point of A.

Open and closed sets

Let's continue with an other classification of a set $A \subseteq \mathbb{R}^n$.

Definition 4.18. Let $A \subseteq \mathbb{R}^n$. Set A is called an open set if all its points are interior, that is $\text{int}(A) = A$.

From the previous definition it is clear that a set is open if it does not have its border. Neighbourhoods are open sets.

Theorem 4.19. Let $\mathbf{x}^0 \in A$ and $\varepsilon > 0$. Then

$$\forall \mathbf{x} \in B_\varepsilon(\mathbf{x}^0),\ \exists \varepsilon_x > 0 \text{ such that } B_{\varepsilon_x}(\mathbf{x}) \subseteq B_\varepsilon(\mathbf{x}^0)$$

Proof. Let $\mathbf{x}$ be any point of $B_\varepsilon(\mathbf{x}^0)$, then $d\left(\mathbf{x}, \mathbf{x}^0\right) < \varepsilon$ and $0 < \varepsilon - d\left(\mathbf{x}, \mathbf{x}^0\right)$. If

$$0 < \varepsilon_x \leq \varepsilon - d\left(\mathbf{x}, \mathbf{x}^0\right)$$

it is clear that

$$\forall \mathbf{y} \in B_{\varepsilon_x}(\mathbf{x}),\ d\left(\mathbf{x}, \mathbf{y}\right) \leq \varepsilon_x \leq \varepsilon - d\left(\mathbf{x}, \mathbf{x}^0\right)$$

But we have

$$d\left(\mathbf{y}, \mathbf{x}^0\right) \leq d\left(\mathbf{x}, \mathbf{x}^0\right) + d\left(\mathbf{x}, \mathbf{y}\right) < d\left(\mathbf{x}, \mathbf{x}^0\right) + \varepsilon - d\left(\mathbf{x}, \mathbf{x}^0\right) = \varepsilon$$

therefore $d\left(\mathbf{y}, \mathbf{x}^0\right) < \varepsilon$ or $y \in B_\varepsilon(\mathbf{x}^0)$.Since we have proved that

$$\forall \mathbf{y} \in B_{\varepsilon_x}(\mathbf{x}),\ y \in B_\varepsilon(\mathbf{x}^0)$$

which is equal to $B_{\varepsilon_x}(\mathbf{x}) \subseteq B_\varepsilon(\mathbf{x}^0)$.

Definition 4.20. Let $A \subseteq \mathbb{R}^n$. The set $A \cup \partial A$, indicated with $\overline{A}$, is called the closure of A ($A \subseteq \overline{A}$).

Definition 4.21. Let $A \subseteq \mathbb{R}^n$. Set A is said to be a closed set if it has all its boundary points (i.e. $A = \overline{A}$).

Example 4.22. Let $A = [0, 10]$ and $B = (0, 10)$. A is a closed set (in fact all its boundary points belong to A) and B is open (in fact all its boundary points do not belong to B or B is only made up of interior points). Let $C = [0, 10] \cup \{20\}$. We observe that $\partial C = \{0, 10, 20\}$, therefore

$$C = C \cup \partial C = \overline{C} \qquad \Rightarrow \qquad C \text{ is closed}$$

Theorem 4.23. A set $A \subseteq \mathbb{R}^n$ is open if and only if its complement is closed.

Corollary 4.24. Set $A \subseteq \mathbb{R}^n$ is closed in and only if it has all its limit points.

Open and closed sets

Remark 4.25. We have $\text{int}(A) \subseteq A \subseteq \overline{A}$.

Corollary 4.26. The boundary of set $A \subseteq \mathbb{R}^n$ is a closed set.

Example 4.27. The set $\mathbb{N}$ of natural numbers is a closed set. Intervals with form $[a, b]$, $[a, +\infty)$ and $(-\infty, b]$ are closed sets (a, b real numbers).

Example 4.28. Consider on $\mathbb{R}^2$ the following subsets

$$A = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_2 = x_1^2 \right\} \quad \text{and} \quad B = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 = 4 \right\}$$

A and B (they are only made up of boundary points) are closed sets.

Theorem 4.29. (a) The intersection of a finite family of open sets is open, while the union of any family (finite or not) of open sets is open. (b) The union of a family of closed sets is closed, while the intersection of any family (finite or not) of closed sets is closed.

From Theorem 4.28 point (a), we have if A_1, A_2, ..., A_n, ... are open sets then

$$\bigcap_{s=1}^{n} A_s \text{ is open} \qquad \bigcup_{s=1}^{n} A_s \text{ is open} \qquad \bigcup_{s=1}^{+\infty} A_s \text{ is open}$$

Compact sets

A set $A \subseteq \mathbb{R}$ which is bounded from above and below is said to be a bounded set.

Definition 4.30. Let $A \subseteq \mathbb{R}^n$. A is said to be bounded if

$$\forall \mathbf{x} \in A,\ \exists k > 0 \text{ such that } \|\mathbf{x}\| < k$$

Definition 4.31. Let $A \subseteq \mathbb{R}^n$. A is said to be a compact set if it is both closed and bounded.

Example 4.32. Consider $A = [2, 3] \cup [5, 7]$. A is closed (it is only made up of interior and boundary points) and bounded (for any $x \in A$, we can consider $k = 8$ such that $|x| < 10$).

Example 4.33. Consider on $\mathbb{R}^2$ the following subsets

$$A = \left\{\mathbf{x} \in \mathbb{R}^2 : x_2 \geq x_1^2\right\} \quad \text{and} \quad B = \left\{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\right\}$$

A and B (they are only made up of interior and boundary points) are closed sets. A is not bounded (it does not exist any k such that a neighbourhood of $\mathbf{0}$ with radius k contains A) and B is bounded (we have $B \subseteq B_k(\mathbf{0})$, for any $k > 4$). Therefore A is not a compact set, while B is a compact set.

Examples

Example 4.34. Consider on $\mathbb{R}^2$ the following subsets

$$A = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_2 \geq x_1^2 \right\} \quad \text{and} \quad B = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4 \right\}$$

Set $A \cap B$ is defined by as follows

$$A \cap B = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_2 \geq x_1^2 \text{ and } x_1^2 + x_2^2 \leq 4 \right\}$$

and set $A \cup B$ is defined as follows

$$A \cup B = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_2 \geq x_1^2 \text{ or } x_1^2 + x_2^2 \leq 4 \right\}$$

$A \cap B$ is closed (intersection of closed sets), bounded (intersection of bounded sets) and compact, while $A \cup B$ is closed, not bounded and not compact. $\mathbf{0} \in A \cap B$ is a boundary point (and limit point) of $A \cap B$. $\mathbf{0} \in A \cup B$ is an interior point (and limit point) of $A \cup B$. We have

$$(A \cap B)^c = A^c \cup B^c = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_2 < x_1^2 \text{ or } x_1^2 + x_2^2 > 4 \right\}$$

which is open and unbounded.