# Mathematics and Statistics (BEMACS) - A. Y. 2019/20

## Lecture 5

BEMACS

Decision Sciences (Bocconi University)

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# Notion of one variable real functions

**Definition 5.1.** Let $A$, $B$ be two sets (of real numbers). A *function* from $A$ to $B$, indicated by $f : A \to B$ ($f$ is defined on $A$ with values on $B$), is a rule of correspondence that assigns to each element of $A$ (input set) exactly one element of $B$ (output set).

Sets $A$ and $B$ are called, respectively, the *domain* and the *codomain* of $f$. Indicating with $x$ any element of $A$ and with $y$ any element of $B$, the $y = f(x)$ is called the image of $x$ under $f$ (while $x$ is called the inverse image or preimage of $y$ under $f$). The set of all possible outputs (or images) coming from all possible inputs of $A$ is called the range (or image) of $A$ under $f$, and it will be denoted by $f(A)$. Therefore we can write

$$f(A) := \{y \in \mathbb{R} : y = f(x), \text{ with } x \in A\} \subseteq B$$

The largest set $A$ in which it is possible to calculate the image of $x$ under $f$ is called the natural domain of $f$. From now on $B = \mathbb{R}$ (i.e. the codomain will be always $\mathbb{R}$).

**Example 5.2.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x-1}$. The natural domain of $f$ is $A = [1, +\infty)$.

# Image and preimage of a set

**Example 5.3.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2 + 1$. The domain of $f$ is $A = \mathbb{R}$ or $A = (-\infty, +\infty)$, while the codomain of $f$ is $\mathbb{R}$ and the range of $f$ is $f(\mathbb{R}) = [1, +\infty)$ (remember that $x^2 \geq 0$ for any $x$ real number). We observe that $f(\mathbb{R})$ is a proper subset of $\mathbb{R}$ (the codomain of $f$), i. e. $f(\mathbb{R}) \subset \mathbb{R}$.

**Definition 5.4.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. (a) If $E \subseteq A$, then the (direct) *image* of $E$ under $f$ is the subset $f(E)$ of $\mathbb{R}$ defined by

$$f(E) := \{y \in \mathbb{R} : y = f(x), \text{ with } x \in E\}$$

(b) If $H \subseteq \mathbb{R}$ (the codomain of $f$), then the *inverse image* of $H$ under $f$ is the subset $f^{-1}(H)$ of $A$ defined by

$$f^{-1}(H) = \{x \in A : y = f(x), \text{ with } y \in H\}$$

**Example 5.5.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2 + 1$. If $E = [-2, 2]$ then the direct image of $E$ under $f$ is defined by

$$f([-2, 2]) := \left\{y \in \mathbb{R} : y = x^2 + 1, \text{ with } x \in [-2, 2]\right\}$$

# Image and preimage of a set

From $x \in [-2, 2]$, a subset of the domain of $f$, we have

$$x \in [-2, 2] \quad \Rightarrow \quad x^2 \in [0, 4] \quad \Rightarrow \quad x^2 + 1 \in [1, 5]$$

and we write $f([-2, 2]) = [1, 5]$.
If $H = [1, 5]$ then the inverse image of $H$ under $f$ is

$$f^{-1}([1, 5]) = \left\{ x \in \mathbb{R} : y = x^2 + 1, \text{ with } y \in [1, 5] \right\}$$

From $y \in [1, 5]$, a subset of the codomain of $f$, we have

$$x^2 + 1 \in [1, 5] \quad \Rightarrow \quad x^2 \in [0, 4] \quad \Rightarrow \quad x \in [-2, 2]$$

and we write $f^{-1}(H) = [-2, 2]$.
Let $y$ be any element of the codomain of $f$. The set of preimages of $y$ is

$$f^{-1}(\{y\}) = \left\{ x \in \mathbb{R} : y = x^2 + 1, \text{ with } y \in \mathbb{R} \right\}$$

where $f^{-1}(\{y\}) = \emptyset$, for any $y < 1$, and $f^{-1}(\{y\}) \neq \emptyset$, for any $y \geq 1$.

# Image and preimage of a set

**Theorem 5.6.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $A_1 \subseteq A$ and $A_2 \subseteq A$. Then

(a) If $A_2 \subseteq A_1$ then $f(A_2) \subseteq f(A_1)$

(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

(c) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(d) $f(A_1) - f(A_2) \subseteq f(A_1 - A_2)$

**Proof.** (a) Let $y \in \mathbb{R}$ be any element of $f(A_2)$. Assume that $A_2 \subseteq A_1$, we have

$$\begin{aligned} y \in f(A_2) \quad &\Rightarrow \quad \exists x \in A_2 : y = f(x) \\ &\Rightarrow \quad \exists x \in A_1 : y = f(x) \Rightarrow y \in f(A_1) \end{aligned}$$

therefore $f(A_2) \subseteq f(A_1)$. (b) Let $y \in \mathbb{R}$ be any element of $f(A_1 \cap A_2)$. We have

$$\begin{aligned} y \quad \in \quad & f(A_1 \cap A_2) \Rightarrow \exists x \in A_1 \cap A_2 : y = f(x) \\ \Rightarrow \quad & \exists x \in A_1 : y = f(x) \text{ and } \exists x \in A_2 : y = f(x) \Rightarrow y \in f(A_1) \text{ and } y \in f(A_2) \end{aligned}$$

from which $y \in f(A_1) \cap f(A_2)$ and we have $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.■

# Image and preimage of a set

**Theorem 5.7.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $A_1 \subseteq \mathbb{R}$ and $A_2 \subseteq \mathbb{R}$. Then

(a) If $A_2 \subseteq A_1$ then $f^{-1}(A_2) \subseteq f^{-1}(A_1)$
(b) $f^{-1}(A_1 \cap A_2) = f^{-1}(A_1) \cap f^{-1}(A_2)$
(c) $f^{-1}(A_1 \cup A_2) = f^{-1}(A_1) \cup f^{-1}(A_2)$
(d) $f^{-1}(A_1) - f^{-1}(A_2) = f^{-1}(A_1 - A_2)$

**Proof.** (a) Let $A_1, A_2$ be subsets of $\mathbb{R}$ such that $A_2 \subseteq A_1$. Consider any element $x$ of $f^{-1}(A_2)$. We have

$$f(x) \in A_2 \quad \Rightarrow \quad f(x) \in A_1 \quad \Rightarrow \quad x \in f^{-1}(A_1)$$

from which we can conclude that $f^{-1}(A_2) \subseteq f^{-1}(A_1)$.■

**Example 5.8.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2 + 1$. Let $A_2 = (5, +\infty)$ and $A_1 = [1, 10]$ be subsets of $\mathbb{R}$, therefore

$$A_1 - A_2 = [1, 10] - (5, +\infty) = [1, 5]$$

We have

$$f^{-1}([1, 10]) - f^{-1}(5, +\infty) = [-2, 2] = f^{-1}([1, 5])$$

# Graph of a function

**Example 5.9.** Let $g : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $g(x) = 1 - \sqrt{x}$. We have $A = [0, +\infty)$ and $g(A) = (-\infty, 1] \subseteq \mathbb{R}$. Let's calculate some images under $g$, we have

$$g(0) = 1 \qquad g(4/9) = 1/3 \qquad g(-4) \text{ does not exist}$$

The inverse image of $H = [-4, 0]$ under $g$ is

$$g^{-1}([-4, 0]) \quad \Rightarrow \quad 1 - \sqrt{x} \in [-4, 0] \quad \Rightarrow \quad \sqrt{x} \in [1, 5]$$

and we get $x \in [1, 25]$.

**Definition 5.10.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. The set of pairs $(x, y)$, indicated with $G_f$, defined by

$$G_f = \{(x, y) : y = f(x), \text{ with } x \in A,\ y \in \mathbb{R}\}$$

is said to be the **graph** of the function $f$ ($G_f$ is a subset of the plane $\mathbb{R}^2$, i.e. $G_f \subseteq \mathbb{R}^2$).

**Remark 5.11.** Not all curves in a plane are graphs of functions (e.g. the relation $x^2 + y^2 = 4$, with $x$, $y$ real numbers, can be represented by a circle with centre $(0, 0)$ and radius $r = 2$, but this graph does not represent a graph of a function).

# Surjective functions

**Definition 5.12.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. A function $f$ is said to be *surjective* on $A$ when every element $y \in \mathbb{R}$ has at least one element (or preimage) $x \in A$ such that $y = f(x)$.

**Example 5.13.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = 2x - 1$. Let $y \in \mathbb{R}$ be any element of the codomain of $f$. We have

$$y = 2x - 1 \quad \Rightarrow \quad 2x = y + 1 \quad \Rightarrow \quad x = \frac{y+1}{2}$$

therefore there exists for any $y$ at least an element $x$ such that $y = f(x)$ (in this case the preimage $x$ exists and is unique).

**Example 5.14.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = 2x^2 + 1$. Let $y \in \mathbb{R}$ be any element of the codomain of $f$, we have

$$y = 2x^2 + 1 \quad \Rightarrow \quad y - 1 = 2x^2 \quad \Rightarrow \quad \frac{y-1}{2} = x^2$$

If $y < 1$, the equation is impossible (the LS is negative and RS is positive or null). We can say that $f$ is not surjective on $\mathbb{R}$.

# Surjective functions

**Remark 5.15.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. (a) A function $f$ is said to be surjective on $A$ when range and codomain are equal, that is $f(A) = \mathbb{R}$ (this strategy is used when is very easy to find the image of $A$ under $f$). (b) From the definition 1.18, we can write the definition of surjective function (in a compact form) as follows

$$\forall y \in \mathbb{R}, \exists x \in A \text{ such that } y = f(x)$$

(c) If $f$ is a surjective function on $A$, we also say that $f$ is a *surjection* on $A$. (d) If $f$ is a surjective function on $A$ then every horizontal line $y = k$, with $k \in \mathbb{R}$, intersects the graph of $f$ in at least one point.

**Example 5.16.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = 1/x$, with natural domain $A = \mathbb{R} \setminus \{0\}$ (in fact $f(0)$ does not exist). Let $y \in \mathbb{R}$ be any element of the codomain of $f$, we have

$$\frac{1}{x} = y \quad \Rightarrow \quad \begin{cases} y = 0 & \nexists x \in A \text{ such that } f(x) = 0 \\ & \\ y \neq 0 & \exists! x \in A \text{ such that } f(x) = y \end{cases}$$

therefore the function $f$ is not surjective on $A$ and its range is $f(A) = \mathbb{R} \setminus \{0\} \subset \mathbb{R}$.

# Injective function

**Definition 5.17.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. A function $f$ is said to be *injective* on $A$ when every element $y \in \mathbb{R}$ has at most one element (or preimage) $x \in A$ such that $y = f(x)$.

**Example 5.18.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = 2x + 1$ ($A = \mathbb{R}$). Let $y \in \mathbb{R}$ be any element of the codomain of $f$. We have

$$y = 2x + 1 \quad \Rightarrow \quad 2x = y - 1 \quad \Rightarrow \quad x = \frac{y-1}{2}$$

therefore there exists for any $y$ at most an element $x$ such that $y = f(x)$ (in this case the preimage $x$ exists and is unique). Therefore $f$ is injective on $\mathbb{R}$.

**Example 5.19.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = 2x^2 + 1$ ($A = \mathbb{R}$). Let $y \in \mathbb{R}$ be any element of the codomain of $f$. We have

$$y = 2x^2 + 1 \quad \Rightarrow \quad 2x^2 = y - 1$$

If $y < 1$ then the preimage $x$ does not exist (under $f$), while if $y \geq 1$ then we obtain one or two preimages $x$ under $f$. Function $f$ is not injective on $\mathbb{R}$.

# Injective functions

**Remark 5.20.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. (a) From the definition 2.1, we can write the definition of an injective function as follows

$$\forall x_1, x_2 \in A, \quad x_1 \neq x_2 \Rightarrow \ f(x_1) \neq f(x_2)$$

(b) If $f$ is an injective function on $A$, we also say that $f$ is an *injection on $A$*. (c) If $f$ is an injective function then every horizontal line $y = k$, with $k \in \mathbb{R}$, intersects the graph of $f$ in at most one point (i.e. horizontal line test). (d) A function $f$ is not injective on $A$ when

$$\exists x_1, x_2 \in A \text{ such that } x_1 \neq x_2 \quad \text{and} \quad f(x_1) = f(x_2)$$

**Example 5.21.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = 2x^2 + 1$ ($A = \mathbb{R}$). Considering $x_1 = -2 \neq x_2 = 2$ we have

$$f(-2) = 2(-2)^2 + 1 = 9 \quad \text{and} \quad f(2) = 2 \cdot 2^2 + 1 = 9 \Rightarrow f(-2) = f(2)$$

form which we say that $f$ is not injective on $\mathbb{R}$.

# Bijective functions

**Definition 5.22.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. A function $f$ is said to be *bijective* on $A$ when every element $y \in \mathbb{R}$ has a unique element (or preimage) $x \in A$ such that $y = f(x)$.

**Exampe 5.23.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = x^3 - 1$ $(A = \mathbb{R})$. Let $y \in \mathbb{R}$ be any element of the codomain of $f$. We have

$$y = x^3 - 1 \quad \Rightarrow \quad x^3 = y + 1 \quad \Rightarrow \quad x = \sqrt[3]{y+1}$$

therefore there exists for any $y$ a unique element or preimage $x$ such that $y = f(x)$. Therefore $f$ is bijective on $\mathbb{R}$.

**Exampe 5.24.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = x^4 + 1$ $(A = \mathbb{R})$. Let $y \in \mathbb{R}$ be any element of the codomain of $f$. We have

$$y = x^4 + 1 \quad \Rightarrow \quad x^4 = y - 1 \quad \Rightarrow \quad \begin{cases} y \geq 1 & x = \sqrt[4]{y-1} \\ y < 1 & \text{Impossible} \end{cases}$$

therefore there exists for any $y$ at most an element or preimage $x$ such that $y = f(x)$. Therefore $f$ is not bijective on $\mathbb{R}$.

# Bijective functions

**Remark 5.25.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. (a) From the definition 5.22, we can write the definition of an bijective function as follows

$$\forall y \in \mathbb{R}, \quad \exists! x \in A \text{ such that } y = f(x)$$

(b) If $f$ is a bijective function on $A$, we also say that $f$ is a b*ijection on A*. (c) $f$ is a bijective function on $A$ if and only if $f$ is injective and surjective function on $A$.

**Example 5.26.** (a) Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = 2x^2$ $(A = \mathbb{R})$. Function $f$ is neither injective nor surjective on $\mathbb{R}$, therefore $f$ is not a bijective function on $\mathbb{R}$. Let $g : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function with $g(x) = \sqrt{x}$ $(A = [0, +\infty))$. (b) Function $g$ on $A$ is injective but is not surjective, therefore $g$ is not a bijective function on $A$. (c) From (b) we have

$$f \text{ is not a bijection on } A \Leftrightarrow f \text{ is not an inj. or } f \text{ is not a surj. on } A$$

**Remark 5.27.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a bijective function on $A$. Then $(\alpha f)(x) + \beta$, with $\alpha \neq 0$, $\beta$ real numbers, is a bijective function on $A$.

# Bounded functions

**Definition 5.28.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Function $f$ is said to be:

1. bounded from above on $A$ if

$$\forall x \in A, \exists M \in \mathbb{R} \text{ such that } f(x) \leq M$$

2. bounded from below on $A$ if

$$\forall x \in A, \exists N \in \mathbb{R} \text{ such that } f(x) \geq N$$

3. bounded on $A$ if

$$\forall x \in A, \exists M, N \in \mathbb{R} \text{ such that } N \leq f(x) \leq M$$

**Example 5.29.** Consider $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ and $g(x) = 1 - 2x^2$. For any $x \in \mathbb{R}$, we have

$$f(x) = x^2 + 1 \geq M = 1 \quad \text{and} \quad g(x) = 1 - 2x^2 \leq N = 1$$

therefore $f$ is bounded from below and $g$ is bounded from above on $\mathbb{R}$.

# Bounded functions

**Remark 5.30.** Let $G_f$ be the graph of $f$. Function $f$ is bounded from above by $M$ means that all the points of $G_f$ have ordinate less than or equal to $M$. Function $f$ is bounded from below by $N$ means that all the points of $G_f$ have ordinate greater than or equal to $N$.

**Theorem 5.31.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Function $f$ is bounded on $A$ if and only if

$$\forall x \in A,\ \exists M > 0 \text{ such that } |f(x)| \leq M$$

**Example 5.32.** Consider $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-x^2}$ and $g(x) = 2\left(x^2+1\right)^{-1}$. We have

$$-1 \leq f(x) \leq 1 \Rightarrow |f(x)| \leq 1 \qquad \text{and} \qquad -2 \leq g(x) \leq 2 \Rightarrow |g(x)| \leq 2$$

and $f$, $g$ are bounded on $\mathbb{R}$.

**Remark 5.33.** Let $f(A)$ be the image of $A$ under $f$. Function $f$ is bounded on $A$ if and only if $f(A)$ is bounded on $\mathbb{R}$.

# Monotonic functions

**Definition 5.34.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Function $f$ is said to be:

1. increasing on $A$ when
$$\forall x_1, x_2 \in A, \quad x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$
2. strictly increasing on $A$ when
$$\forall x_1, x_2 \in A, \quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$
3. decreasing on $A$ when
$$\forall x_1, x_2 \in A, \quad x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$
4. strictly decreasing on $A$ when
$$\forall x_1, x_2 \in A, \quad x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

An (strictly) increasing or (strictly) decreasing function $f$ on $A$ is said to be a (strictly) monotonic function on $A$.

**Example 5.35.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 1$. We have

$$\forall x_1, x_2 \in \mathbb{R}, \quad x_1 < x_2 \;\Rightarrow\; 2x_1 - 1 < 2x_2 - 1 \;\Rightarrow\; f(x_1) < f(x_2)$$

therefore $f$ is strictly increasing on $\mathbb{R}$.

# Monotonic functions

**Remark 5.36.** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. **(a)** A strictly monotonic function on $A$ is also a monotonic function on $A$ (i. e. the set of strictly monotonic functions is a subset of the set of monotonic functions), that is

$$f \text{ strictly monotonic on } A \quad \Rightarrow \quad f \text{ monotonic on } A$$

the opposite is false.
**(b)** If a function $f$ is at the same time increasing and decreasing on $A$ then $f$ is a constant function on $A$ (the inverse implication is also true). We have

$$f \text{ constant on } A \quad \Leftrightarrow \quad f \text{ increasing and decreasing on } A$$

**Example 5.37.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = e^{-x} + 1$. We have

$$\forall x_1, x_2 \in \mathbb{R}, \quad x_1 < x_2 \quad \Rightarrow \quad -e^{x_1} > -e^{x_2} \Rightarrow -e^{x_1} + 1 > -e^{x_2} + 1$$

$$\Rightarrow \quad f(x_1) > f(x_2)$$

therefore $f$ is strictly decreasing on $\mathbb{R}$.

# Composite functions

We often want to combine two functions $f$, $g$ by first finding $f(x)$ and the applying $g$ to $f$ to get $g(f(x))$. However, this is possible only when the image $f(x)$ belongs to the domain of $g$. In order to be able to do this for all $f(x)$, we must assume that the image of $f$ is contained in the domain of $g$.

**Definition 5.37.** Let $f : A_f \subseteq \mathbb{R} \to \mathbb{R}$ and $g : A_g \subseteq \mathbb{R} \to \mathbb{R}$ two functions such that $f(A_f) \subseteq A_g$. The composite function $g \circ f$ ($g$ composite $f$) is the function from $A_f$ to $\mathbb{R}$ such define by

$$(g \circ f)(x) := g[f(x)] \qquad \text{for any } x \in A_f$$

**Example 5.38.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) = x^2$ and $g(x) = 1 - x$. We have

$$f(A_f) = f(\mathbb{R}) = [0, +\infty) \subseteq \mathbb{R} = A_g \qquad g(A_g) = g(\mathbb{R}) = \mathbb{R} = A_f$$

The composite functions $(g \circ f) : \mathbb{R} \to \mathbb{R}$ and $(f \circ g) : \mathbb{R} \to \mathbb{R}$ are defined by

$$g[f(x)] = g\left[x^2\right] = 1 - x^2 \quad \text{and} \quad f[g(x)] = f[1 - x] = (1 - x)^2$$

# Composite functions

**Theorem 5.39.** Let $f : A_f \subseteq \mathbb{R} \to \mathbb{R}$ and $g : A_g \subseteq \mathbb{R} \to \mathbb{R}$ two functions such that $f(A_f) \subseteq A_g$. If $f$ and $g$ are monotons (strictly monotone) on $A_f$ then $g \circ f$ is monotone (strictly monotone) on $A_f$. Moreover:

1. if $g$ and $f$ have the same monotonic property then $g \circ f$ is increasing;
2. if $g$ and $f$ have a different monotonic property then $g \circ f$ is decreasing.

Let $x_1$, $x_2$ be elements on $A_f$ with $x_1 < x_2$. Consider the following table

| | $f$ increasing | $f$ decreasing |
|---|---|---|
| | $f(x_1) \leq f(x_2)$ | $f(x_1) \geq f(x_2)$ |
| $g$ increasing | $(g \circ f)(x_1) \leq (g \circ f)(x_2)$ | $g \circ f(x_1) \geq g \circ f(x_2)$ |
| $g$ decreasing | $g \circ f(x_1) \geq g \circ f(x_2)$ | $g \circ f(x_1) \leq g \circ f(x_2)$ |

**Example 5.40.** Let $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ and $g(x) = x^3$. Functions $f$, $g$ are both increasing on $\mathbb{R}$, therefore $(g \circ f)$ and $(g \circ f)$ are increasing on $\mathbb{R}$.