Mathematics and Statistics (BEMACS) - A. Y. 2019/20

Lecture 6

BEMACS

Decision Sciences (Bocconi University)

18 September 2019

Invertible functions

Definition 6.1. Let $f : A \to B$ be a function with $f(A) \subseteq B$. Function f is said to be an invertible function on A when there is a one-to-one correspondence between A and $f(A)$. We indicate with f^{-1} the inverse function of f and we have

$$f^{-1} : f(A) \to A \quad \text{and} \quad x = f^{-1}(y)$$

Example 6.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 2x + 1$, with $A = \mathbb{R}$ and $f(\mathbb{R}) = \mathbb{R}$. Function f is bijective (injective and bijective) on $\mathbb{R}$, therefore there is a one-to-one correspondence between $\mathbb{R}$ and $f(\mathbb{R})$ and f is invertible on $\mathbb{R}$. We can find $f^{-1} : \mathbb{R} \to \mathbb{R}$ and we have

$$y = 2x + 1 \quad \Rightarrow \quad y - 1 = 2x \quad \Rightarrow \quad x = f^{-1}(y) = \frac{y-1}{2}$$

Example 6.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{2x}$, with $A = [0, +\infty)$ and $f(A) = [0, +\infty)$. Function f is bijective on $[0, +\infty)$, therefore there is a one-to-one correspondence between $[0, +\infty)$ and $f(A) = [0, +\infty)$ and f is invertible on $[0, +\infty)$. We can write that $f^{-1} : [0, +\infty) \to [0, +\infty)$ and we have

$$y = \sqrt{2x} \quad \Rightarrow \quad y^2 = 2x \quad \Rightarrow \quad x = f^{-1}(y) = y^2/2$$

Invertible functions

Theorem 6.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If f is an injective function on A if and only if f is an invertible function on A.

From theorem 6.4 we have: (a) If f is an injective function on A then f is an invertible function on A; (b) If f is an invertible function on A then f is injective on A.

Example 6.5. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 3x + 5$, with $A = \mathbb{R}$ and $f(\mathbb{R}) = \mathbb{R}$. Since f is invertible on $\mathbb{R}$, function f is invertible and hence we can define the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$.

Remark 6.6. Consider an invertible function $f : A \subseteq \mathbb{R} \to \mathbb{R}$. (a) $y = f^{-1}(x) \Leftrightarrow x = f(y)$. (b) The domain of f is the image of f. (c) The image of f^{-1} is the domain of f.

Property 6.7. Let's consider an invertible function $f : A \subseteq \mathbb{R} \to \mathbb{R}$. We have

$$\forall x \in A,\ f^{-1}\left[f(x)\right] = x \qquad \text{and} \qquad \forall y \in f(A),\ f\left(f^{-1}(y)\right) = y$$

The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

Examples

Example 6.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \left(x^3 - 1\right)^3 + 1$, with $f(\mathbb{R}) = \mathbb{R}$. Function f is surjective and injective on $\mathbb{R}$ (composition of injective functions is also an injective one). Therefore f is invertible on $\mathbb{R}$ and $f^{-1} : \mathbb{R} \to \mathbb{R}$. We have

$$y = \left(x^3 - 1\right)^3 + 1 \quad \Rightarrow \quad y - 1 = \left(x^3 - 1\right)^3 \quad \Rightarrow \quad x^3 - 1 = \sqrt[3]{y - 1}$$

and

$$x^3 = 1 + \sqrt[3]{y - 1} \quad \Rightarrow \quad x = f^{-1}(y) = \sqrt[3]{1 + \sqrt[3]{y - 1}}$$

Let's check if the inverse of f just found is right, we have

$$\begin{aligned} f^{-1}\left[f(x)\right] &= \sqrt[3]{1 + \sqrt[3]{f(x) - 1}} = \sqrt[3]{1 + \sqrt[3]{\left(x^3 - 1\right)^3 + 1 - 1}} \\ &= \sqrt[3]{1 + \sqrt[3]{\left(x^3 - 1\right)^3}} = \sqrt[3]{1 + x^3 - 1} = \sqrt[3]{x^3} = x \qquad \text{for any } x \in \mathbb{R} \end{aligned}$$

Global extrema

Definition 6.9. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Point $x^* \in A$ is said to be:

1. a (global) maximizer of f on A if

$$\forall x \in A, \quad f(x) \leq f(x^*)$$

   The value $f(x^*)$ is called the maximum value of f on A.

2. a (global) minimizer of f on A if

$$\forall x \in A, \quad f(x^*) \leq f(x)$$

   The value $f(x^*)$ is called the minimum value of f on A.

Example 6.10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 1 - \left(x^2 + 1\right)^3$. Point $x^* = 0$ is a (global) maximizer of f on $\mathbb{R}$ because

$$\forall x \in \mathbb{R}, \quad f(x) = 1 - \left(x^2 + 1\right)^3 \leq 0 = f(0)$$

and 0 is the maximum value of f on $\mathbb{R}$.

Optimization problems

Remark 6.11. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. A (global) maximizer $x^* \in A$ of f on A is said to be strong if

$$\forall x \in A - \{x^*\}, \quad f(x) < f(x^*)$$

and a (global) minimizer of f on A is said to be strong if

$$\forall x \in A - \{x^*\}, \quad f(x^*) < f(x)$$

A (global) maximizer or minimizer of f in A is strong if and only if it is unique.

Example 6.12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 1 + |5x|$. Point $x^* = 0$ is a strong (global) minimizer of f on $\mathbb{R}$ because

$$\forall x \in \mathbb{R} - \{x^*\}, \quad f(x^*) = 1 < 1 + |5x| = f(x)$$

and 1 is the minimum value of f on $\mathbb{R}$.

Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that we have an optimization problem when

$$\max_{x \in A} f(x) \qquad \text{or} \qquad \min_{x \in A} f(x)$$

i.e. we want to find the maximum or minimum value, if they exist, of the function f on A.

Equivalent optimization problems

Theorem 6.13. Let $g : A_g \subseteq \mathbb{R} \to \mathbb{R}$ be a strictly increasing function with $f(A_f) \subseteq A_g$. The two optimization problems

$$\max_{x \in A} f(x) \qquad \text{and} \qquad \max_{x \in A} (g \circ f)(x)$$

are equivalent (they have the same maximizer).
Proof. Let $x^* \in A$ be a (global) maximizer of f, therefore for any $x \in A$ we have $f(x) \leq f(x^*)$. Since g is strictly increasing on A, we have

$$f(x) \leq f(x^*) \quad \Rightarrow \quad (g \circ f)(x) \leq (g \circ f)(x^*)$$

and $x^* \in A$ is also a (global) maximizer of $(g \circ f)$.

Example 6.14. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 1 - x^2$. Point $x^* = 0$ is a strong (global) maximizer of f on $\mathbb{R}$ (with maximum value 1). Consider $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x^3$, therefore g is strictly increasing and $(g \circ f)$ exists. Then

$$(g \circ f)(x) = g[f(x)] = g\left[1 - x^2\right] = 2\left(1 - x^2\right)^3$$

and $x^* = 0$ is still a strong (global) maximizer of $(g \circ f)$ on $\mathbb{R}$ (with maximum value 2). We have the same if we consider the following functions

Local extrema

Definition 6.15. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Point $x^* \in A$ is said to be:

1. a local maximizer of f on A if $B_\varepsilon(x^*)$ exists such that

$$\forall x \in A \cap B_\varepsilon(x^*), \quad f(x) \leq f(x^*)$$

   The value $f(x^*)$ is called the local maximum value of f on A.

2. a local minimizer of f on A if $B_\varepsilon(x^*)$ exists such that

$$\forall x \in A \cap B_\varepsilon(x^*), \quad f(x^*) \leq f(x)$$

   The value $f(x^*)$ is called the local minimum value of f on A.

Remark 6.16. A (global) maximizer or minimizer of f on A is also a local maximizer/minimizer of f on A (the opposite is false).

Example 6.17. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} |x| & x \neq 0 \\ 2 & x = 0 \end{cases}$$

where $x = 0$ is a local maximizer of f on $\mathbb{R}$ with $f(0) = 2$.

Local extrema

In fact, it is possible to take a radius $\varepsilon > 0$, e.g. $\varepsilon = 1/2$, such that

$$\forall x \in \mathbb{R} \cap B_\varepsilon(0), \quad f(x) \leq f(0) = 2$$

It is obvious that this property is lost if we compare $f(0)$ with $f(x)$ using any $x \in \mathbb{R}$.

Example 6.18. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} -x^2 & x \neq 0 \\ -2 & x = 0 \end{cases}$$

where $x = 0$ is a local minimizer of f on $\mathbb{R}$ with $f(0) = -2$. In fact, it is possible to take a radius $\varepsilon > 0$, e.g. $\varepsilon = 1/2$, such that

$$\forall x \in \mathbb{R} \cap B_\varepsilon(0), \quad f(0) = -2 \leq f(x)$$

It is obvious that this property is lost if we compare $f(0)$ with $f(x)$ using any $x \in \mathbb{R}$.

Example 6.19. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} -x^2 & x \neq 0 \\ 2 & x = 0 \end{cases}$$

where $x = 0$ is a global maximizer of f on $\mathbb{R}$ with $f(0) = 2$.

Concave and convex functions

Definition 6.20. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and A an interval. Function f is said to be:

1. a concave function on A if

$$f\left[\alpha x_1 + (1-\alpha)\, x_2\right] \geq \alpha f(x_1) + (1-\alpha)\, f(x_2)$$

   for any $x_1, x_2 \in A$ and $\alpha \in [0, 1]$.

2. a convex function on A if

$$f\left[\alpha x_1 + (1-\alpha)\, x_2\right] \leq \alpha f(x_1) + (1-\alpha)\, f(x_2)$$

   for any $x_1, x_2 \in A$ and $\alpha \in [0, 1]$.

Remark 6.21. A function is concave on A if the segment between points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on G_f is below G_f (we have the opposite for a convex function).

Example 6.22. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = |x|$. We have

$$f\left[\alpha x_1 + (1-\alpha)\, x_2\right] = \left|\alpha x_1 + (1-\alpha)\, x_2\right| \leq \alpha f(x_1) + (1-\alpha)\, f(x_2)$$

therefore f is convex on $\mathbb{R}$.

Strictly concave and strictly convex functions

Definition 6.23. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and A an interval. Function f is said to be:

1. a strictly concave function on A if

$$f\left[\alpha x_1 + (1-\alpha)\, x_2\right] > \alpha f(x_1) + (1-\alpha)\, f(x_2)$$

   for any $x_1, x_2 \in A$, $x_1 \neq x_2$ and $\alpha \in (0,1)$.

2. a convex function on A if

$$f\left[\alpha x_1 + (1-\alpha)\, x_2\right] < \alpha f(x_1) + (1-\alpha)\, f(x_2)$$

   for any $x_1, x_2 \in A$, $x_1 \neq x_2$ and $\alpha \in (0,1)$.

Remark 6.24. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$, with a, b real numbers, is said to be an affine function on $\mathbb{R}$. We have

$$f\left[\alpha x_1 + (1-\alpha)\, x_2\right] = a\left(\alpha x_1 + (1-\alpha)\, x_2\right) + b = \alpha f(x_1) + (1-\alpha)\, f(x_2)$$

therefore f is both convex and concave on $\mathbb{R}$.