Mathematics and Statistics (BEMACS) - A. Y. 2019/20

Lecture 7

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Limits of functions

Definition 7.1. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \overline{\mathbb{R}}$ a limit point of A. We write

$$\lim_{x \to x_0} f(x) = L \in \overline{\mathbb{R}} \qquad \text{or} \qquad f(x) \to L \in \overline{\mathbb{R}} \text{ as } x \to x_0$$

if, for any neighbourhood $V_\varepsilon(L)$, a neighbourhood $U_{\delta_\varepsilon}(x_0)$ exists such that

$$x_0 \neq x \in U_{\delta_\varepsilon}(x_0) \cap A \Rightarrow f(x) \in V_\varepsilon(L)$$

and L is said to be the value of the limit of f at x_0.

We have four cases: (a) x_0, $L \in \mathbb{R}$; (b) $x_0 \in \mathbb{R}$, $L = \pm\infty$; (c) $x_0 = \pm\infty$, $L \in \mathbb{R}$; (d) $x_0 = \pm\infty$, $L = \pm\infty$.

Example 7.2. Let $f : (0, +\infty) \to \mathbb{R}$ and $x_0 = +\infty$ a limit point of A. We write $f(x) \to +\infty$ as $x \to 0$ if for any $V_\varepsilon(+\infty)$ a neighbourhood $U_{\delta_\varepsilon}(0)$ exists such that

$$0 \neq x \in U_{\delta_\varepsilon}(0) \cap (0, +\infty) \Rightarrow f(x) \in V_\varepsilon(+\infty)$$

Let $f : (0, +\infty) \to \mathbb{R}$ and $x_0 = 5$ a limit point of A. We write $f(x) \to 2$ as $x \to 5$ if for any $V_\varepsilon(5)$ a neighbourhood $U_{\delta_\varepsilon}(2)$ exists such that

$$2 \neq x \in U_{\delta_\varepsilon}(2) \cap (0, +\infty) \Rightarrow f(x) \in V_\varepsilon(5)$$

One-sided limits

Definition 7.3. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ a limit point of A. We write

$$\lim_{x \to x_0^+} f(x) = L \in \overline{\mathbb{R}} \qquad \text{or} \qquad f(x) \to L \in \overline{\mathbb{R}} \text{ as } x \to x_0^+$$

if, for any neighbourhood $V_\varepsilon(L)$, a right neighbourhood $U^+_{\delta_\varepsilon}(x_0)$ exists such that

$$x_0 \neq x \in U^+_{\delta_\varepsilon}(x_0) \cap A \Rightarrow f(x) \in V_\varepsilon(L)$$

and L is said to be the value of the right limit of f at x_0.

Example 7.4. Let $f : (0, +\infty) \to \mathbb{R}$ and $x_0 = 0$ a limit point of A. We write $f(x) \to 0$ as $x \to 0^+$ if, for any $V_\varepsilon(0)$, a right neighbourhood $U^+_{\delta_\varepsilon}(0)$ exists such that $0 \neq x \in U^+_{\delta_\varepsilon}(0) \cap (0, +\infty) \Rightarrow f(x) \in V_\varepsilon(0)$ and 0 is the value of the limit of f at 0. Let $f : (0, +\infty) \to \mathbb{R}$ and $x_0 = 0$ a limit point of A. We write $f(x) \to +\infty$ as $x \to 0^+$ if, for any $V_\varepsilon(+\infty)$, a right neighbourhood $U^+_{\delta_\varepsilon}(0)$ exists such that

$$0 \neq x \in U^+_{\delta_\varepsilon}(0) \cap (0, +\infty) \Rightarrow f(x) \in V_\varepsilon(+\infty)$$

and $+\infty$ is the value of the limit of f at 0 (we say that f diverges to $+\infty$ as x converges to 0 from the right).

One-sided limits

Definition 7.5. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ a limit point of A. We write

$$\lim_{x \to x_0^-} f(x) = L \in \overline{\mathbb{R}} \qquad \text{or} \qquad f(x) \to L \in \overline{\mathbb{R}} \text{ as } x \to x_0^-$$

if, for any neighbourhood $V_\varepsilon(L)$, a left neighbourhood $U^-_{\delta_\varepsilon}(x_0)$ exists such that

$$x_0 \neq x \in U^-_{\delta_\varepsilon}(x_0) \cap A \Rightarrow f(x) \in V_\varepsilon(L)$$

and L is said to be the value of the left limit of f at x_0.

Example 7.6. Let $f : (-\infty, 0) \to \mathbb{R}$ and $x_0 = 0$ a limit point of A. We write $f(x) \to 0$ as $x \to 0^-$ if, for any $V_\varepsilon(0)$, a left neighbourhood $U^-_{\delta_\varepsilon}(0)$ exists such that $0 \neq x \in U^-_{\delta_\varepsilon}(0) \cap (-\infty, 0) \Rightarrow f(x) \in V_\varepsilon(0)$ and 0 is the value of the limit of f at 0. Let $f : (-\infty, 0) \to \mathbb{R}$ and $x_0 = 0$ a limit point of A. We write $f(x) \to -\infty$ as $x \to 0^-$ if, for any $V_\varepsilon(-\infty)$, a left neighbourhood $U^-_{\delta_\varepsilon}(0)$ exists such that

$$0 \neq x \in U^-_{\delta_\varepsilon}(0) \cap (0, +\infty) \Rightarrow f(x) \in V_\varepsilon(-\infty)$$

and $-\infty$ is the value of the limit of f at 0 (we say that f diverges to $-\infty$ as x converges to 0 from the left).

Limit theorems

Theorem 7.7. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that a $U_{\delta_\varepsilon}(x_0)$ exists such that $U_{\delta_\varepsilon}(0) - \{x_0\} \subseteq A$. Then

$$\lim_{x \to x_0} f(x) = L \in \overline{\mathbb{R}} \quad \Leftrightarrow \quad \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L \in \overline{\mathbb{R}}$$

Example 7.8. Let $f, g : \mathbb{R} \to \mathbb{R}$ be a functions defined by

$$f(x) = \begin{cases} 1/|x| & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1/x^3 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We have

$$\lim_{x \to 0^+} \frac{1}{|x|} = \lim_{x \to 0^-} \frac{1}{|x|} = +\infty \quad \Leftrightarrow \quad \lim_{x \to 0} f(x) = +\infty$$

and

$$\lim_{x \to 0^+} \frac{1}{x^3} = +\infty \neq -\infty = \lim_{x \to 0^-} \frac{1}{x^3} \quad \text{and} \quad \nexists \lim_{x \to 0} g(x)$$

that is the limit of g as $x \to 0$ (x converges to 0 from the left and from the right) does not exist (while the two one-sided limits exist).

Limit theorems

Theorem 7.9. (**Uniqueness of the limit**). Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ a limit point of A. At most a unique $L \in \overline{\mathbb{R}}$ exists such that

$$\lim_{x \to x_0} f(x) = L$$

Proof (by contradiction). Suppose that two different limits L_1 and L_2 exist, $L_1 \neq L_2$. For any $\varepsilon > 0$, we have:

- there exists $\delta_1(\varepsilon/2)$ such that

$$x_0 \neq x \in U_{\delta_1(\varepsilon/2)}(x_0) \cap A \Rightarrow |f(x) - L_1| < \varepsilon/2$$

- there exists $\delta_2(\varepsilon/2)$ such that

$$x_0 \neq x \in U_{\delta_2(\varepsilon/2)}(x_0) \cap A \Rightarrow |f(x) - L_2| < \varepsilon/2$$

Let $\delta = \inf(\delta_1(\varepsilon/2), \delta_2(\varepsilon/2))$. Therefore a $\delta(\varepsilon/2)$ exists such that

$$x_0 \neq x \in U_{\delta(\varepsilon/2)}(x_0) \cap A \Rightarrow |f(x) - L_1| < \varepsilon/2 \text{ and } |f(x) - L_2| < \varepsilon/2$$

From the triangle inequality, we have

$$|L_2 - L_1| < \varepsilon \quad \Rightarrow \quad L_2 - L_1 = 0 \quad \Rightarrow \quad L_2 = L_1$$

therefore if the limit exists, it is unique.

Limit theorems

Theorem 7.10 (Permanence of the sign). Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ a limit point of A. If $\lim_{x \to x_0} f(x) = L \neq 0$, then a neighbourhood $B_\varepsilon(x_0)$ exists from which f and L have the same sign, that is

$$\forall x \in (B_\varepsilon(x_0) - \{x_0\}) \cap A, \quad f(x)L > 0$$

Theorem 7.11 (Comparison criterion). Let $f, g, h : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ a limit point of A. If

$$\forall x \in A,\ g(x) \leq f(x) \leq h(x) \quad \text{and} \quad \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L$$

then $\lim_{x \to x_0} f(x) = L$.

Example 7.12. Show that $\lim_{x \to 0} \left[x^2 \cos(1/x)\right] = 0$. For any $x \in \mathbb{R}$ and $x \neq 0$, we have

$$\left|x^2 \cos(1/x)\right| \leq \left|x^2\right| \cdot \left|\cos(1/x)\right| \leq \left|x^2\right| = x^2$$

therefore $-x^2 \leq x^2 \cos(1/x) \leq x^2$ and $-x^2, x^2 \to 0$ as $x \to 0$. Therefore $\lim_{x \to 0} \left[x^2 \cos(1/x)\right] = 0$ (comparison criterion).

Algebra of limits

Theorem 7.13. Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ a limit point of A. If

$$\lim_{x \to x_0} f(x) = L \in \overline{\mathbb{R}} \quad \text{and} \quad \lim_{x \to x_0} g(x) = M \in \overline{\mathbb{R}}$$

Then:

1. the limit of $(f+g)(x)$ exists and
$$\lim_{x \to x_0} (f+g)(x) = L + M \in \overline{\mathbb{R}}$$
provided that $L + M$ is not an indeterminate, that is $+\infty - \infty$ and $-\infty + \infty$.
2. the limit of $(fg)(x)$ exists and
$$\lim_{x \to x_0} (fg)(x) = L \cdot M \in \overline{\mathbb{R}}$$
provided that LM is not an indeterminate, that is $\pm\infty \cdot 0$ and $0 \pm \infty$.
3. if a neighbourhood of x_0 exists, with $x \neq x_0$, then the limit of $(f/g)(x)$ exists and
$$\lim_{x \to x_0} (f/g)(x) = L/M \in \overline{\mathbb{R}}$$
provided that L/M is not an indeterminate, that is $\pm\infty / \pm\infty$ and $0/0$.

Horizontal, vertical and oblique asymptotes

Definition 7.14. If at least one of the conditions

$$\lim_{x \to x_0^+} f(x) = \pm\infty \qquad ; \qquad \lim_{x \to x_0^-} f(x) = \pm\infty$$

holds, the line $x = x_0 \in \mathbb{R}$ is called a vertical asymptote of (the graph of) $f(x)$. If at least one of the conditions

$$\lim_{x \to +\infty} f(x) = y_0 \qquad ; \qquad \lim_{x \to -\infty} f(x) = y_0$$

holds, the line $y = y_0 \in \mathbb{R}$ is called a horizontal asymptote of (the graph of) $f(x)$.

Definition 7.15. Let a, b real numbers. If

$$\lim_{x \to +\infty} (f(x) - ax - b) \qquad ; \qquad \lim_{x \to -\infty} (f(x) - ax - b)$$

that is if the distance between f and the straight line $y = ax + b$ goes to 0 as $x \to +\infty / -\infty$, then the straight line is said to be an oblique asymptote of f as x diverges.

Limits of elementary functions

- Power function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$, with $n \in \mathbb{N}$. We have

$$\lim_{x \to x_0} x^n = x_0^n \quad ; \quad \lim_{x \to +\infty} x^n = +\infty \quad ; \quad \lim_{x \to -\infty} x^n = \begin{cases} +\infty & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$$

- Power function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(x) = x^\alpha$, with $\alpha \in \mathbb{R}$. We have

$$\lim_{x \to 0^+} x^\alpha = \begin{cases} +\infty & \alpha < 0 \\ 1 & \alpha = 0 \\ 0 & \alpha > 0 \end{cases} \quad \text{and} \quad \lim_{x \to +\infty} x^\alpha = \begin{cases} 0 & \alpha < 0 \\ 1 & \alpha = 0 \\ +\infty & \alpha > 0 \end{cases}$$

- Exponential function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \alpha^x$, with $0 < \alpha$. For any x_0 real number, we have $\alpha^x \to \alpha^{x_0}$ when $x \to x_0$. If x goes to $+\infty$ or $-\infty$, we have

$$\lim_{x \to +\infty} \alpha^x = \begin{cases} 0 & \alpha < 1 \\ 1 & \alpha = 1 \\ +\infty & \alpha > 1 \end{cases} \quad \text{and} \quad \lim_{x \to -\infty} \alpha^x = \begin{cases} +\infty & \alpha < 1 \\ 1 & \alpha = 1 \\ 0 & \alpha > 1 \end{cases}$$

Limits of elementary functions

- Logarithmic function $f : (0, +\infty) \longrightarrow \mathbb{R}$ defined by $f(x) = \log_\alpha x$, with $0 < \alpha \neq 1$. For any $x_0 > 0$ real number, we have $\log_\alpha x \longrightarrow \log_\alpha x_0$ when $x \longrightarrow x_0$. If x goes to $+\infty$ or 0 (from the right), we have

$$\lim_{x \to +\infty} \log_\alpha x = \begin{cases} -\infty & \alpha < 1 \\ +\infty & \alpha > 1 \end{cases} \quad \text{and} \quad \lim_{x \to 0^+} \log_\alpha x = \begin{cases} +\infty & \alpha < 1 \\ -\infty & \alpha > 1 \end{cases}$$

- Trigonometric functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \sin x$ and $f(x) = \cos x$. For any x_0 real number, we have

$$\lim_{x \to x_0} \sin x = \sin x_0 \quad \text{and} \quad \lim_{x \to x_0} \cos x = \cos x_0$$

and limits of $\sin x$ and $\cos x$ don't exist as x diverges to $+\infty$ or $-\infty$.

Example 7.16. Consider $f(x) = e^x$ and $g(x) = \ln x$. We have

$$\lim_{x \to 0^+} \frac{\ln x}{e^x} = +\infty \ ; \ \lim_{x \to 0^+} \frac{e^x}{\ln x} = 0 \ ; \ \lim_{x \to +\infty} e^x \ln x = +\infty \ ; \ \lim_{x \to +\infty} \frac{1}{e^x \ln x} = 0$$

and

$$\lim_{x \to +\infty} \left(\frac{1}{e^x} + \ln x \right) = +\infty \ ; \ \lim_{x \to +\infty} \left(\frac{1}{e^x} + \frac{1}{\ln x} \right) = 0 \ ; \ \lim_{x \to +\infty} \left(\frac{1}{\ln x} - e^x \right) = -\infty$$

Remarkable limits

- We have

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0 \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

- $f(x)\to\pm\infty$ as $x\to x_0$, then for $x\to x_0$

$$\left(1+\frac{k}{f(x)}\right)^{f(x)}\to e^k \quad (\text{with } k\in\mathbb{R})$$

- $f(x)\to 0$ as $x\to x_0$, then for $x\to x_0$

$$\frac{a^{f(x)}-1}{f(x)}\to\log a \quad (\text{with } a>1)$$

Example 7.17. We have

$$\lim_{x\to 0}\frac{x^2\sin x}{(1-\cos x)\,x}=2 \quad \lim_{x\to 0}\left(1-\frac{1}{2x^2}\right)^{x^2}=\sqrt{e} \quad \lim_{x\to 1}\frac{2^{(x-1)^2}-1}{(x-1)^2}=\log 2$$

Remarkable limits

- $f(x) \longrightarrow 0$ as $x \longrightarrow x_0$, then for $x \longrightarrow x_0$

$$\frac{\log_a(1+f(x))}{f(x)} \longrightarrow \frac{1}{\log a} \quad (\text{with } 0 < a \neq 1)$$

- $f(x) \longrightarrow 0$ as $x \longrightarrow x_0$, then for $x \longrightarrow x_0$

$$\frac{(1+f(x))^{\alpha}-1}{f(x)} \longrightarrow \alpha$$

Example 7.18. We have

$$\lim_{x\to+\infty} \frac{\log_2(1+1/x^2)}{1/x^2} = \frac{1}{\log 2} \quad \lim_{x\to 0} \frac{\left(1+x^2\right)^3-1}{x^2} = 3$$

and

$$\lim_{x\to 0} \frac{\left(1+2x^2\right)^{-1/3}-1}{2x^2} = -1/3$$

Orders of convergence and divergence

Definition 7.19. Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ a limit point of A for which $B_\varepsilon(x_0)$ exists such that $g(x) \neq 0$ for any $x \in A \cap B_\varepsilon(x_0)$.

1. If $(f/g)(x) \to 0$ as $x \to x_0$ then f is said to be negligible with respect to g as $x \to x_0$. We write $f = o(g)$ (which is read "f is a litte-o of g") as $x \to x_0$.
2. If $(f/g)(x) \to k \neq 0$ as $x \to x_0$ then f is said to be comparable with respect to g as $x \to x_0$. We write $f \asymp g$ (which is read "f is comparable with g") as $x \to x_0$. If
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$
then f, g are said to be asymptotic functions as $x \to x_0$. We write $f \sim g$ as $x \to x_0$.

Remark 7.20. (a) The negligibility relation is transitive, that is

$$f = o(g) \quad \text{and} \quad g = o(h) \quad x \to x_0 \quad \Rightarrow \quad f = o(h) \quad x \to x_0$$

(b) The asymptotic relation is symmetric and transitive.

Example 7.21. $\sqrt{x}/x \to 0$ as $x \to +\infty$, therefore $\sqrt{x} = o(x)$ as $x \to +\infty$.

Little-o algebra

Theorem 7.22. Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ and c a non-zero real number. Then

(a) $o(f) + o(f) = o(f)$ (b) $o(f)o(g) = o(fg)$ (c) $c \cdot o(f) = o(cf) = o(f)$

(d) $o(g) + o(f) = o(f)$ if $g = o(f)$.

Theorem 7.23. Let $k, h > 0$, $\alpha > 1$ and $a > 1$. Then:

1. $x^k = o\left(\alpha^x\right)$ as $x \to +\infty$;
2. $x^h = o\left(x^k\right)$ as $x \to +\infty$ and $h < k$;
3. $\log_a x = o\left(x^k\right)$ as $x \to +\infty$.

From the transitivity property we have $\log_a x = o\left(\alpha^x\right)$ as $x \to +\infty$.

Example 7.24. For $x \to +\infty$, we have

$$x^2 = o\left(e^x\right) \qquad x^3 = o\left(x^5\right) \qquad \log_4 x = o\left(\sqrt{x}\right) \qquad \ln x = o\left(e^x\right)$$

Remark 7.25. If $f \to 0$ as $x \to x_0$ then $f = o(1)$ as $x \to x_0$.

Asymptotic equivalence

If $f(x) \sim g(x)$ as $x \longrightarrow x_0$, then

$$\lim_{x \longrightarrow x_0} f(x) = L \Leftrightarrow \lim_{x \longrightarrow x_0} g(x) = L$$

that is f, g have the same limit $L \in \mathbb{R}$ as $x \longrightarrow x_0$.

Theorem 7.26. Let $f(x) \sim g(x)$ and $h(x) \sim l(x)$ as $x \longrightarrow x_0$. Then:

- $f(x) \cdot h(x) \sim g(x) \cdot l(x)$ as $x \longrightarrow x_0$.
- $f(x)/h(x) \sim g(x)/l(x)$ as $x \longrightarrow x_0$, provided that $h(x) \neq 0$ and $l(x) \neq 0$ for any $x \neq x_0$ of $B_\varepsilon(x_0)$.

Example 7.27. For $x \longrightarrow +\infty$, we have

$$\frac{x^2 + \ln x + e^{-x}}{x^3 + e^{2x} + 10} \sim \frac{x^2}{e^{2x}} \longrightarrow 0 \qquad \frac{x^4 + x + e^{-2x}}{x^3 + e^{-2x} + 10} \sim \frac{x^4}{x^3} \longrightarrow +\infty$$

and

$$\frac{3x^3\left(e^{1/x} - 1\right)}{2x^2} \sim \frac{3x^3 \cdot (1/x)}{2x^2} = \frac{3x^3}{2x^3} \longrightarrow \frac{3}{2}$$

Continuous functions

Definition 7.28. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in A$ a limit point of A. Function f is said to be continuous at x_0 when

$$\lim_{x \to x_0} f(x) = f(x_0)$$

and f is continuos at each isolated point of A.

A function f continuous at every point of a subset E of A is said to be continuous on E. The set of all the continuous functions on set E is indicated with $C(E)$. Elementary functions are continuous (on their domains A). If a function f is not continuous at x_0 then we say that f has a discontinuity point at x_0.

Example 7.29. Let $f : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$f(x) = \begin{cases} 1/|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Function f is continuous at any point $x_0 \neq 0$ because we have

$$\lim_{x \to x_0} \frac{1}{|x|} = \frac{1}{|x_0|} = f(0)$$

Examples

We observe that

$$\lim_{x \to 0^-} \frac{1}{|x|} = \lim_{x \to 0^+} \frac{1}{|x|} = +\infty \quad \text{and} \quad f(0) = 0$$

therefore f is not continuous at $x_0 = 0$ (f has limit, as x goes to 0, from the left and from the right).

Example 7.30. Let $f : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$f(x) = \begin{cases} 1/x^3 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Function f is continuous at any point $x_0 \neq 0$ because we have

$$\lim_{x \to x_0} \frac{1}{x^3} = \frac{1}{x^3} = f(0)$$

We observe that

$$\lim_{x \to 0^-} \frac{1}{x^3} = -\infty \neq +\infty = \lim_{x \to 0^+} \frac{1}{x^3} \quad \text{and} \quad f(0) = 0$$

therefore f is not continuous at $x_0 = 0$.

Discontinuity points

Definition 7.31. Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in A$ a limit point of A. If

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L \neq L^* = f(x_0)$$

we say that f has a removable discontinuity point at x_0. If

$$\lim_{x \to x_0^-} f(x) = \left\{ \begin{array}{c} \pm\infty \\ \nexists \end{array} \right. \quad \text{or} \quad \lim_{x \to x_0^+} f(x) = \left\{ \begin{array}{c} \pm\infty \\ \nexists \end{array} \right.$$

we say that f has a non-removable discontinuity point at x_0.

Example 7.32. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \left\{ \begin{array}{ll} 1/x & x \geq 1 \\ e^x & x < 1 \end{array} \right.$$

Function f at $x = 1$ is not continuous, in fact we have

$$\lim_{x \to 1^-} e^x = e \qquad \text{and} \qquad \lim_{x \to 1^+} \frac{1}{x} = 1$$

and we say that f at $x = 1$ has a jump (discontinuity point).

Properties of continuous functions

Theorem 7.33. Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be continuous functions at $x_0 \in A$. Then

1. $f + g$ is continuous at $x_0 \in A$.
2. fg is continuous at $x_0 \in A$.
3. f/g is continuos at $x_0 \in A$, provided that $g(x_0) \neq 0$.

Theorem 7.34. Let $f : A_f \subseteq \mathbb{R} \to \mathbb{R}$ and $g : A_g \subseteq \mathbb{R} \to \mathbb{R}$ be functions such that $f(A_f) \subseteq A_g$. If f is continuous at $x_0 \in A$ and g is continuous at $f(x_0)$, then $(g \circ f)$ is continuous at $x_0 \in A$.

Theorem 7.35. (Bolzano). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)f(b) \leq 0$ then at least a point $c \in [a, b]$ exists such that $f(c) = 0$. If f is strictly monotonic on $[a, b]$ then point c is unique.

Example 7.36. Let $f : [0, 3] \to \mathbb{R}$ be a continuous function defined by $f(x) = x^2 - 2$. Since $f(0)f(3) \leq 0$, function f has at least a point $c \in [0, 3]$ such that $f(c) = 0$. Let $g : [0, 1] \to \mathbb{R}$ be a continuous function defined by

$$g(x) = x^3 + x - 1$$

Since $g(0)g(1) \leq 0$, function g has at least a point $c \in [0, 3]$ such that $g(c) = 0$. In this case, g is monotonic and point c is unique.

Properties of continuous functions

Theorem 7.37. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. If $f(a) \geq g(a)$ and $f(b) \leq g(b)$ then a point $c \in [a, b]$ exists such that $f(c) = g(c)$. If f is strictly increasing and g is strictly decreasing, then point c is unique.

Theorem 7.38. (Weierstrass). A continuous function $f : [a, b] \to \mathbb{R}$ has at least a minimizer and at least a maximizer on $[a, b]$.

Example 7.39. Let $g : [0, 1] \to \mathbb{R}$ be a continuous function defined by

$$g(x) = x^3 + x - 1$$

therefore g has at least a minimizer and maximizer on $[0, 1]$ (0 and 1).

Theorem 7.40. (Intermediate value theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous functions. Set

$$m = \min_{x \in [a,b]} f(x) \qquad \text{and} \qquad M = \max_{x \in [a,b]} f(x)$$

then for any $z \in [m, M]$ a point c exists such that $f(c) = z$. If f is strictly monotonic on $[a, b]$ then point c is unique.

Theorem 7.41. Let $f : I \to \mathbb{R}$ be a continuous function on interval I. Function f is injective if and only if f is strictly monotonic.