Mathematics and Statistics (BEMACS) - A. Y. 2019/20

Lecture 8

BEMACS

Decision Sciences (Bocconi University)

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Notion of derivative of a function

Let $f : (a,b) \to \mathbb{R}$ be a function and $x_0 \in (a,b)$, the ratio defined as follows

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{h}$$

is called the difference quotient of f with respect to $[x_0, x_0+h]$, with $h \neq 0$ and $x_0 + h \in (a,b)$.

Remark 8.1. If $x = x_0 + h$, with $h \neq 0$, then the difference quotient of f can be written as follows

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

Example 8.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x + x^2$. The difference quotient of f with respect to $[x_0, x_0+h]$, with $h \neq 0$, is

$$\begin{aligned} \frac{\Delta f}{\Delta x} &= \frac{\left[(x_0+h) + (x_0+h)^2\right] - \left(x_0 + x_0^2\right)}{h} = \frac{(x_0+h)^2 - x_0^2}{h} \\ &= \frac{h + h(2x_0+h)}{h} = \frac{h\left[1 + (2x_0+h)\right]}{h} = 1 + (2x_0+h) \end{aligned}$$

Notion of derivative of a function

Definition 8.3. Let $f : (a, b) \to \mathbb{R}$ be a function and $x_0 \in (a, b)$. Function f is said to be derivable at x_0 if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and is finite. The value of the limit, indicated with $f'(x_0)$, is called the derivative of f at x_0.

Example 8.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x + x^2$. The limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} [1 + (2x_0 + h)] = 1 + 2x_0$$

exists and is finite, therefore f is derivable at x_0 and $f'(x_0) = 1 + 2x_0$.

Example 8.5. Let $f : (0, +\infty) \to \mathbb{R}$ be a function defined by $f(x) = \ln x$. The limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \left[\frac{\ln(x_0 + h) - \ln x_0}{h} \right] = \lim_{h \to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x_0}\right) = \frac{1}{x_0}$$

exists and is finite, therefore f is derivable at $x_0 > 0$ and $f'(x_0) = 1/x_0$.

Geometric interpretation

Definition 8.6. Let $f : (a, b) \to \mathbb{R}$ be a derivable function at $x_0 \in (a, b)$. The equation defined by

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is called the equation of the tangent line to the graph of the function f at x_0. The slope of the tangent line is $f'(x_0)$.

Example 8.7. Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) = 3x^2 + 2$. Function g is derivable at x_0 with $g'(x_0) = 6x_0$. Therefore the equation of the tangent line to the graph of the function g at x_0 is

$$y = g(x_0) + g'(x_0)(x - x_0)$$

and we get

$$y = 3x_0^2 + 2 + 6x_0(x - x_0) = 6x_0x - 3x_0^2 + 2$$

Remark 8.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a constant function defined by $f(x) = k$. Since $f'(x_0) = 0$, for any $x_0 \in \mathbb{R}$, the equation of the tangent line to the graph of the function f at x_0 is $y = f(x_0) = k$ (which is equal to function f).

Derivative function

Let $f : (a, b) \to \mathbb{R}$ be a function. The set $D \subseteq (a, b)$ where f is derivable is said to be the domain of derivability of f.

Example 8.9. Let $g : [0, +\infty) \to \mathbb{R}$ be a function defined by $g(x) = \sqrt{x}$. We have

$$\lim_{h\to 0} \frac{g(x_0 + h) - g(x_0)}{h} = \lim_{h\to 0} \left[\frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h}\right] = \lim_{h\to 0} \left[\frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}}\right]$$

For any $x_0 > 0$, we have

$$\lim_{h\to 0} \left[\frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}}\right] = \frac{1}{2\sqrt{x_0}}$$

and g is derivable, but at $x_0 = 0$ the limit does not exist and g is not derivable at this point. The domain of derivability of g is $D = (0, +\infty) \subseteq [0, +\infty)$.

Definition 8.10. Let $f : (a, b) \to \mathbb{R}$ be a function with $D \subseteq (a, b)$. The function $f' : D \to \mathbb{R}$ which associates at each $x \in D$ the derivative $f'(x)$ is said to be the derivative of function of f.

One-sided derivatives

Definition 8.11. Let $f : (a, b) \to \mathbb{R}$ be a function. Function f is said to be derivable from right and from the left at $x_0 \in (a, b)$ if the one-sided limits

$$f'_+(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{and} \quad f'_-(x_0) = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h}$$

exist and are finite.

Theorem 8.12. Let $f : (a, b) \to \mathbb{R}$ be a function. Function f is derivable at $x_0 \in (a, b)$ if and only if $f'_+(x_0) = f'_-(x_0)$. We have

$$f'(x_0) = f'_+(x_0) = f'_-(x_0)$$

Example 8.13. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = e^x$. We have

$$f'_+(x_0) = \lim_{h \to 0^+} \frac{e^{x_0+h} - e^{x_0}}{h} = e^{x_0} \quad \text{and} \quad f'_-(x_0) = \lim_{h \to 0^-} \frac{e^{x_0+h} - e^{x_0}}{h} = e^{x_0}$$

therefore $f'_+(x_0) = f'_-(x_0) = e^{x_0}$ and f is derivable at x_0 with $f'(x_0) = e^{x_0}$.

Derivability of a function at a point

There are some case where function f is not derivable at x_0. We have:

- f is derivable at x_0 from the right and from the left but we have
$$f'_{+}(x_0) \neq f'_{-}(x_0) \qquad \text{and} \qquad f'(x_0) \text{ does not exist}$$
and we say that f has a corner point at x_0;
- the difference quotient of f as $x \longrightarrow x_0$, from the left and from the right, is not finite
$$f'_{+}(x_0) = \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \pm\infty$$
and
$$f'_{-}(x_0) = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = \mp\infty$$
and we say that f has a cuspidal point at x_0;

Example 8.14. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $f(x) = \sqrt{|x|}$. At $x_0 = 0$, we have

$$f'_{+}(0) = \lim_{h \to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = +\infty \quad \text{and} \quad f'_{-}(0) = \lim_{h \to 0^-} \frac{\sqrt{0+h} - \sqrt{0}}{h} = -\infty$$

therefore f is not derivable at 0 (we have a cuspidal point).

Derivability of a function at a point

- the difference quotient of f as $x \longrightarrow x_0$, from the left and from the right, is

$$f'_+(x_0) = \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \pm\infty$$

and

$$f'_-(x_0) = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = \pm\infty$$

and we say that f has a vertical inflection point at x_0.

Example 8.15. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $f(x) = |x|$. At $x_0 = 0$, we have

$$f'_+(0) = \lim_{h \to 0^+} \frac{(0+h) - 0}{h} = 1 \quad \text{and} \quad f'_-(x_0) = \lim_{h \to 0^-} \frac{(0+h) - 0}{h} = -1$$

with $f'_+(0) = 1 \neq -1 = f'_-(0)$ and f is not derivable at 0 (we have a corner point).

Example 8.16. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \sqrt[3]{x}$. At $x_0 = 0$, we have

$$f'_+(0) = \lim_{h \to 0^+} \frac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h} = +\infty \quad \text{and} \quad f'_-(0) = \lim_{h \to 0^-} \frac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h} = +\infty$$

therefore f is not derivable at 0 (we have a vertical inflection point).

Derivability and continuity

Theorem 8.17. Let $f : (a, b) \to \mathbb{R}$ be a derivable function at $x_0 \in (a, b)$. Then f is continuous at $x_0 \in (a, b)$.

Proof. We assume that f is derivable at $x_0 \in (a, b)$, therefore the limit of the difference quotient of f at $x_0 \in (a, b)$ exists and is finite. We have

$$\begin{aligned}
\lim_{x \to x_0} [f(x) - f(x_0)] &= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right] \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot \lim_{x \to x_0} (x - x_0) = 0
\end{aligned}$$

therefore

$$\lim_{x \to x_0} [f(x) - f(x_0)] = 0 \quad \Rightarrow \quad \lim_{x \to x_0} f(x) - f(x_0) = 0 \quad \Rightarrow \quad \lim_{x \to x_0} f(x) = f(x_0)$$

and f is continuous at $x_0 \in (a, b)$.

The converse of theorem 8.17 is false, that is the continuity of f at $x_0 \in (a, b)$ does not imply the derivability at the same point.

Derivability and continuity

Example 8.18. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = |x|$. Function f is continuous on $\mathbb{R}$. We observe that at $x_0 = 0$ we have

$$f'_+(0) = 1 \neq -1 = f'_-(0) \text{ (and } f'(x_0) \text{ does not exist)}$$

we can say that f is not derivable at 0 (there is a corner point).

Example 8.19. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with $f(x) = \sqrt{|x|}$. Function f is continuous on $\mathbb{R}$ (in fact $f(x) \to \sqrt{|x_0|}$ as $x \to x_0$ and $f(x_0) = \sqrt{|x_0|}$). We observe that at $x_0 = 0$ we have

$$f'_+(0) = \lim_{h \to 0^+} \frac{\sqrt{|0+h|} - 0}{h} = \lim_{h \to 0^+} \frac{\sqrt{h}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = +\infty$$

and

$$f'_-(0) = \lim_{h \to 0^+} \frac{\sqrt{|0+h|} - 0}{h} = \lim_{h \to 0^+} \frac{\sqrt{-h}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{-h}} = +\infty$$

we can say that function f is not derivable at 0 (there is a cuspidal point).

Derivatives of elementary functions

- Let $f:(0,+\infty)\to\mathbb{R}$ be defined by $f(x)=x^\alpha$, $\alpha\in\mathbb{R}$. Function f is derivable on $(0,\infty)$ and
$$f'(x)=(x^\alpha)'=\alpha x^{\alpha-1}$$
- Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=a^x$, with $a>0$, $a\neq 1$. Function f is derivable on $\mathbb{R}$ and
$$f'(x)=(a^x)'=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=a^x\lim_{h\to 0}\frac{a^h-1}{h}=a^x\ln a$$
If $a=e$, then $f'(x)=(e^x)'=e^x\ln e=e^x$.
- Let $f:(0,+\infty)\to\mathbb{R}$ be defined by $f(x)=\log_a x$, with $a>0$, $a\neq 1$. Function f is derivable on $(0,\infty)$ and
$$\begin{aligned} f'(x) &= (\log_a x)'=\lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\log_a\left(\frac{x+h}{x}\right)=\lim_{h\to 0}\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)=\frac{1}{x\cdot\ln a}\end{aligned}$$
If $a=e$, then $f'(x)=(\ln x)'=1/x$.

Derivative of elementary functions

Example 8.20. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \sqrt[4]{\frac{1}{x^3}}$$

which is derivable $(0, +\infty)$. We have

$$f'(x) = \left(\sqrt[4]{\frac{1}{x^3}}\right)' = \left(x^{-3/4}\right)' = -\frac{3}{4}x^{-7/4} = -\frac{3}{4x^2\sqrt[4]{x^3}}$$

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$. Function f is derivable on $\mathbb{R}$ and

$$f'(x) = (\sin x)' = \cos x$$

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos x$. Function f is derivable on $\mathbb{R}$ and

$$f'(x) = (\cos x)' = -\sin x$$

Example 8.21. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 3^{-x}$. Function f is derivable $\mathbb{R}$. We have

$$f'(x) = \left(3^{-x}\right)' = \left[(1/3)^x\right]' = (1/3)^x \ln 3^{-1} = -3^{-x}\ln 3$$

Algebra of derivatives

Property 8.22. Let f, g be two derivable functions on $A \subseteq \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. We have:

1. $(\alpha f)(x)$ is derivable on A and
$$(\alpha f)'(x) = \alpha \cdot f'(x)$$
2. $(f+g)(x)$ and $(f-g)(x)$ are derivable on A and
$$(f+g)'(x) = f'(x) + g'(x) \qquad (f-g)'(x) = f'(x) - g'(x)$$
3. $(fg)(x)$ is derivable on A and
$$(fg)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
4. $(f/g)(x)$ is derivable on A, with $g(x) \neq 0$, and
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Algebra of derivatives

Rules (1)-(4) are called the constant multiple rule, sum and difference rules, product rule and quotient rule, respectively.

Proof. (a) Suppose f derivable at $x_0 \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. We have

$$(\alpha f)'(x) = \lim_{x \to x_0} \frac{(\alpha f)(x) - (\alpha f)(x_0)}{x - x_0} = \alpha \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \alpha f'(x)$$

(b) Suppose f and g derivable functions at $x_0 \in \mathbb{R}$. We have

$$\begin{aligned}
(f+g)'(x) &= \lim_{x \to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x) + g(x) - f(x_0) - g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right] \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f'(x) + g'(x)
\end{aligned}$$

Examples

Example 8.23. Consider $f(x) = x^3$ and $g(x) = e^x$, both derivable functions on $\mathbb{R}$. Function $(fg)(x)$ is also a derivable function on $\mathbb{R}$ and its derivative is

$$\begin{aligned}(fg)'(x) &= \left(x^3 e^x\right)' = \left(x^3\right)' \cdot e^x + x^3 \cdot (e^x)' \\ &= 3x^2 \cdot e^x + x^3 \cdot e^x = x^2 e^x (3 + x)\end{aligned}$$

Example 8.24. Consider $f(x) = 2x$ and $g(x) = \ln 2x$, both derivable functions on $(0, +\infty)$. Function $(f/g)(x)$ is a derivable function on $(0, +\infty)$ with $x \neq 1$, and its derivative is

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \left(\frac{2x}{\ln 2x}\right)' = \frac{(2x)' \cdot \ln 2x - x \cdot (\ln 2x)'}{(\ln 2x)^2} = \frac{2 \cdot \ln 2x - 2x \cdot (2/2x)}{(\ln 2x)^2} \\ &= \frac{2(\ln 2x - 1)}{(\ln 2x)^2} \qquad \textbf{NB.}\ (\ln 2x)^2 = \ln 2x \cdot \ln 2x \neq \ln^2(2x)\end{aligned}$$

Examples

Example 8.25. Consider $f(x) = kx$ and $g(x) = \ln kx$, with $k > 0$, both derivable functions on $(0, +\infty)$. Function $(g/f)(x)$ is a derivable function on $(0, +\infty)$, and its derivative is

$$\left(\frac{g}{f}\right)'(x) = \left(\frac{\ln kx}{kx}\right)' = \frac{(\ln kx)' \cdot kx - \ln kx \cdot (kx)'}{(kx)^2} = \frac{1 - k \ln kx}{(kx)^2}$$

$$\left(\frac{g}{f}\right)'(x) = \frac{1 - k \ln kx}{(kx)^2} = 0 \Rightarrow x = k^{-1} \sqrt[k]{e} \text{ (stationary point of } (g/f)(x))$$

Example 8.26. Consider $f(x) = x^\alpha$ and $g(x) = e^x$, both derivable functions on $(0, +\infty)$. Function $(f/g)(x)$ is a derivable function on $(0, +\infty)$ with $e^x \neq 0$, and its derivative is

$$\left(\frac{f}{g}\right)'(x) = \left(\frac{x^\alpha}{e^x}\right)' = \frac{(x^\alpha)' \cdot e^x - x^\alpha \cdot (e^x)'}{(e^x)^2} = \frac{\alpha x^{\alpha-1} e^x - x^\alpha e^x}{e^{2x}}$$

$$\left(\frac{f}{g}\right)'(x) = \frac{x^{\alpha-1}(\alpha - x)}{e^x} = 0 \Rightarrow x = \alpha \text{ (stationary point of } (f/g)(x))$$