Mathematics and Statistics (BEMACS) - A. Y. 2019/20

Lecture 9

BEMACS

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Derivative of composite functions

We now turn to the theorem on the derivative of composite functions known as the "Chain Rule". It provides a formula for finding the derivative of a composite function $g \circ f$ in terms of the derivatives of g and f.

Theorem 9.1. Let $f : A_f \subseteq \mathbb{R} \to \mathbb{R}$ and $g : A_g \subseteq \mathbb{R} \to \mathbb{R}$ be functions with $f(A_f) \subseteq A_g$ (A_f, A_g are open subsets of $\mathbb{R}$). If f is derivable at $x \in A_f$ and g is derivable at $f(x) \in A_g$, then the composite function $(g \circ f) : A_f \subseteq \mathbb{R} \to \mathbb{R}$ is derivable at $x \in A_f$ and

$$(g \circ f)'(x) = g'[f(x)] \cdot f'(x)$$

Example 9.2. Let $f, g \subseteq \mathbb{R} \to \mathbb{R}$ be derivables functions with $f(x) = e^x$ and $g(x) = x^2$. Given that $f'(x) = e^x$ and $g'(x) = 2x$, we have:

- $(g \circ f)$ is derivable on $\mathbb{R}$ and its derivative (using the chain rule) is

$$(g \circ f)'(x) = g'[f(x)] \cdot f'(x) = g'[e^x] \cdot e^x = 2(e^x) \cdot e^x = 2e^{2x}$$

- $(f \circ g)$ is derivable on $\mathbb{R}$ and its derivative (using the chain rule) is

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x) = f'\left[x^2\right] \cdot 2x = 2xe^{x^2}$$

Derivative of composite functions

Example 9.3. Let $f, g \subseteq (0, +\infty) \to \mathbb{R}$ be derivable functions with $f(x) = \ln x$ and $g(x) = x^2 + 1$. Given that $f'(x) = 1/x$ and $g'(x) = 2x$, we have:

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x) = f'\left[x^2 + 1\right] \cdot 2x = \frac{2x}{x^2 + 1}$$

Example 9.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For any $x \neq 0$, applying the chain rule, we have

$$\begin{aligned} f'(x) &= \left[x^2 \sin(1/x)\right]' = \left(x^2\right)' \cdot \sin(1/x) + x^2 \cdot [\sin(1/x)]' \\ &= 2x \sin(1/x) + x^2 \cos(1/x) \cdot (1/x)' = 2x \sin(1/x) - \cos(1/x) \end{aligned}$$

and if $x = 0$ we have $f'(0) = 0$ (using the definition of derivative). The derivative f' of f exists for any $x \in \mathbb{R}$. However, f' does not have a limit as $x \to 0$, therefore f' is not continuous at 0.

Examples

Remark 9.5. Under the assumptions of derivability of the composite function $(g \circ f \circ h)(x)$, we have

$$(g \circ f \circ h)(x) = g'[(f \circ h)(x)] \cdot f'[h(x)] \cdot h'(x)$$

Example 9.6. Let $f, g, h \subseteq \mathbb{R} \to \mathbb{R}$ be derivable functions with $f(x) = e^x$, $g(x) = x^2$ and $h(x) = 2x + 1$. Given that $f'(x) = e^x$, $g'(x) = 2x$ and $h'(x) = 2$, we have:

- $(g \circ f \circ h)$ is derivable on $\mathbb{R}$ and its derivative is

$$\begin{aligned}(g \circ f \circ h)(x) &= g'[(f \circ h)(x)] \cdot f'[h(x)] \cdot h'(x) \\ &= 2(f \circ h)(x) \cdot e^2 \cdot 2 = 4e^2 e^{2x+1} = 4e^{2(2x+1)}\end{aligned}$$

- $(f \circ g \circ h)$ is derivable on $\mathbb{R}$ and its derivative is

$$\begin{aligned}(f \circ g \circ h)'(x) &= f'[(g \circ h)(x)] \cdot g'(h(x)) \cdot h'(x) \\ &= e^{(g \circ h)(x)} \cdot [h(x)]^2 \cdot 2 = 2(2x+1)^2 e^{(2x+1)^2}\end{aligned}$$

Derivative of inverse functions

Remark 9.7. Let $f : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous one-to-one (i.e. injective) function on (a, b). Then f^{-1} is continuous on its domain.

If f is derivable on an interval A and is one-to-one on that interval (i.e. injective on A), than for x_0 in A the point (x_0, y_0), where $y_0 = f(x_0)$, on the graph of f and the point (y_0, x_0) on the graph of f^{-1} are mirror images of each other in the line $y = x$ (i.e. the bisector line of I and III quadrants). The slopes of the tangent lines at (x_0, y_0) and (y_0, x_0) are also related. If the graph of f has a horizontal tangent at (x_0, y_0), then the graph of f^{-1} has a vertical tangent at (y_0, x_0).

Theorem 9.8. Let $f : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be a derivable injective function on $x_0 \in (a, b)$. If $f'(x_0) \neq 0$, the inverse function of f (i.e. f^{-1}) is derivable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}y} f^{-1}(y_0) = \frac{1}{\mathrm{d}f(x_0)/\mathrm{d}x}$$

If (x_0, y_0) is a known point of the graph of f, the above formula enables us to calculate the derivative of f^{-1} at (y_0, x_0) without an equation that defines $f^{-1}(y)$.

Derivative of an inverse function

Example 9.9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^3 + x$. Function f is derivable and injective for any $x \in \mathbb{R}$. The derivative of f is

$$f'(x) = \left(x^3 + x\right)' = 3x^2 + 1$$

- Let $x_0 = -1$. Given that $f'(-1) = 4 \neq 0$, the inverse function of f is derivable at $y_0 = 0 = f(-1)$ and

$$\left(f^{-1}\right)'(0) = \frac{1}{f'(-1)} = \frac{1}{4}$$

- Let $x_0 = 0$. Given that $f'(0) = 1 \neq 0$, the inverse function of f is derivable at $y_0 = 0 = f(0)$ and

$$\left(f^{-1}\right)'(0) = \frac{1}{f'(0)} = 1$$

- Let $x_0 = 1$. Given that $f'(1) = 4 \neq 0$, the inverse function of f is derivable at $y_0 = 2 = f(1)$ and

$$\left(f^{-1}\right)'(2) = \frac{1}{f'(1)} = \frac{1}{4}$$

Differentiability of a function

Definition 9.10. Let $f : (a, b) \longrightarrow \mathbb{R}$ be a function. Function f is said to be differentiable at $x_0 \in (a, b)$ if a linear function $l : \mathbb{R} \longrightarrow \mathbb{R}$ exists such that

$$f(x_0 + h) = f(x_0) + l(h) + o(h) \qquad \text{as } h \longrightarrow 0$$

for any $x_0 + h \in (a, b)$.

Remark 9.11. Function f is differentiable at x_0 if the linear function $l : \mathbb{R} \longrightarrow \mathbb{R}$ approximate the difference

$$\Delta f(x) = f(x_0 + h) - f(x_0)$$

with an error of $o(h)$ as $h \longrightarrow 0$ (where $o(h)$ goes to 0 faster than h). The linear function $l : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be the differential of f at $x0$ and is indicated by $df(x_0) : \mathbb{R} \longrightarrow \mathbb{R}$. Therefore we have

$$f(x_0 + h) = f(x_0) + \mathrm{d}f(x_0)(h) + o(h) \qquad \text{as } h \longrightarrow 0$$

Function $f : (a, b) \longrightarrow \mathbb{R}$ is said to be differentiable on (a, b) if it is differentiable at any point $x_0 \in (a, b)$.

Differentiability of a function

Theorem 9.12. Let $f : (a, b) \longrightarrow \mathbb{R}$ be a function. Function f is differentiable at $x_0 \in (a, b)$ if and only if f is derivable on (a, b). In this case, the differential $df(x_0) : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$df(x_0)(h) = f'(x_0) \cdot h$$

Remark 9.13. Under the assumption that f is differentiable at $x_0 \in (a, b)$, we have

$$\Delta f(x) = df(x_0)(h) + o(h) \qquad \text{and} \qquad \Delta f(x) \approx df(x_0)(h)$$

as $h \longrightarrow 0$ (i.e. we can approximate Δf with the differential $df(x_0)(h)$ with $h \longrightarrow 0$).

Theorem 9.14. Let $f : (a, b) \longrightarrow \mathbb{R}$ be a differentiable function at $x_0 \in (a, b)$. Then f is continuous at $x_0 \in (a, b)$.

Proof. Assuming that f is differentiable at $x_0 \in (a, b)$, we have

$$\lim_{h \longrightarrow 0} f(x_0 + h) = \lim_{h \longrightarrow 0} \left[f(x_0) + f'(x_0) \cdot h + o(h) \right] = f(x_0)$$

from which function f is continuous at $x_0 \in (a, b)$.

Derivatives of higher orders

If $f' : D \to \mathbb{R}$ is continuous on $E \subseteq D$, we say that f is continuously differentiable on E. We indicate with $C(E)$ the set of all the continuously differentiable functions on E.

Derivatives of order grater than one are obtained by a natural extension of the derivation process. Let $f' : D \to \mathbb{R}$ the derivative function and $x_0 \in D$ an interior point of D (the domain of derivability of f). We say that f' is derivable at $x_0 \in D$, indicated by $f''(x_0)$, when

$$f''(x_0) = \lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0)}{h}$$

exists and is finite, and we say that f is derivable two times at $x_0 \in D \subseteq A$.

Example 9.15. Let $f : (0, +\infty) \to \mathbb{R}$ defined by $f(x) = \alpha \ln kx^n$, with k, α, $n > 0$. Function f is derivable two times at $x_0 \in (0, +\infty)$ and we have

$$f'(x) = (\alpha \ln kx^n)' = \frac{\alpha n}{x} \qquad \text{and} \qquad f''(x) = \left(\frac{\alpha n}{x}\right)' = -\frac{\alpha n}{x^2}$$

In similar fashion we define the third derivative $f'''(x_0) = f^{(3)}(x_0)$, ..., and the n-th derivative $f^{(n)}(x_0)$, whenever these derivatives exist.

Examples

Example 9.16. Let $f : (0, +\infty) \to \mathbb{R}$ defined by $f(x) = x^m \ln x$, $m > 0$. Function f is derivable on $(0, +\infty)$ and its derivative is

$$\begin{aligned} f'(x) &= (x^m \ln x)' = (x^m)' \cdot \ln x + x^m \cdot (\ln x)' = \\ &= m x^{m-1} \ln x + x^m \frac{1}{x} = m x^{m-1} \ln x + x^{m-1} = x^{m-1} (m \ln x + 1) \end{aligned}$$

The second derivative of f is

$$\begin{aligned} f''(x) &= \left[x^{m-1} (m \ln x + 1) \right]' = \left(x^{m-1} \right)' (m \ln x + 1) + x^{m-1} (m \ln x + 1)' \\ &= x^{m-2} \cdot \left[\left(m^2 - m \right) \ln x + 2m - 1 \right] \end{aligned}$$