30407, Mathematics 2 BEMACS
Problem Set 1
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1 Limits and Continuity

Exercise 1.1 (10 points). *Let* $A = \{(x, y) : x + y \neq 0\}$. *Let* $f : A \to \mathbb{R}$ *be defined by*

$$f(x, y) = \frac{x - y}{x + y}.$$

Find the values of the following limits (note they are nested one-variable limits)

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x, y)\right), \quad \lim_{y\to 0}\left(\lim_{x\to 0} f(x, y)\right).$$

Does

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

exist?

1.1 *For what concerns the first two nested one-variable limits, we have that:*

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x, y)\right) = \lim_{x\to 0}(\frac{x}{x}) = \lim_{x\to 0}(1) = 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x, y)\right) = \lim_{y\to 0}(\frac{-y}{y}) = \lim_{y\to 0}(-1) = -1$$

The different values of the two limits show that, when approaching the origin, the function is not continuous.

In a more general case, approaching the origin from any straight line in $\mathbb{R}^2$*:*

$$y = mx, m \in \mathbb{R}$$

If we take the limit for:

$$(x, mx) \to (0,0)$$

we can see that the limit does not exist. Indeed:

$$\lim_{(x,xm)\to(0,0)} f(x,xm) = \lim_{(x,xm)\to(0,0)} \frac{x-mx}{x+mx} = \lim_{(x,xm)\to(0,0)} \frac{x(1-m)}{x(1+m)} = \lim_{(x,xm)\to(0,0)} \frac{1-m}{1+m}$$

Here we have shown that by approaching the limit from a generic line, we don't find a unique limit, but rather an infinite number of them depending on the value of m.

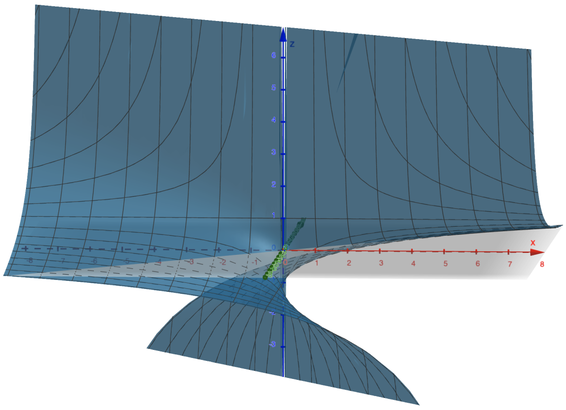

Figure 1: f(x,y) $= \frac{x-y}{x+y}$

Exercise 1.2 (10 points)**.** *Let $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f(x,y) = x^2 + y^2 + 4x + 1, \quad g(x,y) = x - y$$

and

$$h(x,y) = \begin{cases} f(x,y) & y > 2x \\ g(x,y) & y \leq 2x. \end{cases}$$

Find where h is continuous.

1.2 *The two functions: f(x,y) and g(x,y) are everywhere continuous on their domain, they do not have any discontinuity point. Hence, the piece-wise defined function h(x,y), composed of f(x,y) and g(x,y), is continuous in any* *point (x,y) with $y \neq 2x$. It is instead necessary to study the function's behaviour for $y = 2x$.*

By definition the function h(x,y) is continuous in (x, y) if the value of f(x,y) and g(x,y) are equal at the point. Moreover, we can directly use f(x,y) and g(x,y) instead of using the limits as they approach $y = 2x$ because f(x,y) and g(x,y) are continuous on their natural domain.

To evaluate the h function continuity, we consider the following system:

- *The first condition is* $\boldsymbol{y = 2x}$
- *The second condition is* ***f(x,y) = g(x,y)***

$$\begin{cases} y = 2x \\ f(x,y) = g(x,y) \end{cases} \rightarrow \begin{cases} y = 2x \\ x^2 + y^2 + 4x + 1 = x - y \end{cases} \rightarrow \begin{cases} y = 2x \\ 5x^2 + 5x + 1 = 0 \end{cases}$$

After some calculations we obtain a second-degree equation in x. The solutions of this equation are two values of x , substituting them back in $y = 2x$, we obtain two values of y.

$$\begin{cases} y = 2x \\ x_{1,2} = \frac{-5\pm\sqrt{5}}{10} \end{cases} \rightarrow \begin{cases} y_{1,2} = \frac{-5\pm\sqrt{5}}{5} \\ x_{1,2} = \frac{-5\pm\sqrt{5}}{10} \end{cases}$$

We identify two points:

$$Point_1 : \begin{cases} x_1 = \frac{-5+\sqrt{5}}{10} \\ y_1 = \frac{-5+\sqrt{5}}{5} \end{cases} = (\frac{-5+\sqrt{5}}{10}, \frac{-5+\sqrt{5}}{5})$$

$$Point_2 : \begin{cases} x_2 = \frac{-5-\sqrt{5}}{10} \\ y_2 = \frac{-5-\sqrt{5}}{5} \end{cases} = (\frac{-5-\sqrt{5}}{10}, \frac{-5-\sqrt{5}}{5})$$

So, we can conclude that the function h is continuous in any point (x,y) with $y \neq 2x$. While, for $y = 2x$ it is continuous in the two points we have just found.

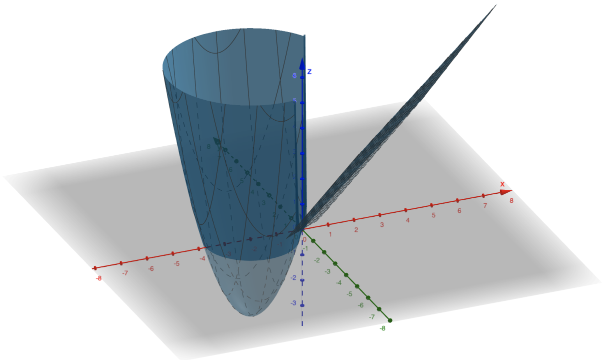

Figure 2: h(x,y) $= \begin{cases} x^2 + y^2 + 4x + 1 & \text{if } \ y > 2x \\ x - y & \text{if } \ y \leq 2x \end{cases}$

2 Linear algebra

Exercise 2.1 (10 points)**.** *Consider* $\mathbb{R}^3$ *and the three vectors* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *and* $\begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}$, *where* $a, b, c \in \mathbb{R}$. *Prove that* a, b, c *are distinct if and only if the three vectors are linearly independent.*

2.1 *In order to prove that a,b,c are distinct if and only if the three vectors are linearly independent, we have to prove the two sides of the statement:*

- *If the three vectors are linearly independent then a,b,c are distinct.*
- *If a,b,c are distinct then the three vectors are linearly independent.*

Three vectors are linearly independent $\rightarrow$ ***a,b,c are distinct.***
We can express the three vectors in the form of a matrix:

$$A = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

According to ***Remark 3.39****: For a square* $(n \times n)$ *matrix* A *(*A *is* 3×3*) the following statements are equivalent:*

- $det(A) \neq 0$
- $rank(A) = n$
- $\exists\ A^{-1}$
- *The columns (the rows) of A are linearly independent (they form a basis).*

By hypothesis, the three vectors that compose the matrix A are linearly independent, so the following inequality holds: $\mathbf{Det(A) \neq 0}$.

$$Det\begin{pmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{pmatrix} \neq 0$$

$$1 \times Det\begin{pmatrix}b & b^2\\ c & c^2\end{pmatrix} - a \times Det\begin{pmatrix}1 & b^2\\ 1 & c^2\end{pmatrix} + a^2 \times Det\begin{pmatrix}1 & b\\ 1 & c\end{pmatrix} \neq 0$$

After some calculations:

$$b \times c^2 - c \times b^2 - a \times c^2 + a \times b^2 + a^2 \times c - a^2 \times b \neq 0$$

Finally, we can reduce the inequality to the following form:

$$-(a-b)(b-c)(c-a) \neq 0$$

Hence, we derive that the only possible combinations of a,b,c satisfying the inequality are those such that:

$$a \neq b \wedge a \neq c \wedge b \neq c$$

Therefore, the linear independence of the three vectors, implies that a,b,c are distinct. ***Q.E.D.***

a,b,c are distinct $\rightarrow$ ***Three vectors are linearly independent.***
We want now to prove that if a,b,c are distinct then the three vectors are linearly independent.

$$Det\begin{pmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{pmatrix} =$$

$$= 1 \times Det\begin{pmatrix}b & b^2\\ c & c^2\end{pmatrix} - a \times Det\begin{pmatrix}1 & b^2\\ 1 & c^2\end{pmatrix} + a^2 \times Det\begin{pmatrix}1 & b\\ 1 & c\end{pmatrix} =$$

$$= b \times c^2 - c \times b^2 - a \times c^2 + a \times b^2 + a^2 \times c - a^2 \times b =$$

$$= -(a-b)(b-c)(c-a)$$

By hypothesis, we know that a,b,c are distinct. Hence, the last term of the equation $\neq 0$*, which implies* $\mathbf{Det(A)} \neq \mathbf{0}$.

Therefore, referring again to ***Remark 3.39*** *we have shown that the three vectors are linearly independent.* ***Q.E.D.***

Exercise 2.2 (5 points)**.** *Prove or disprove: Let* $S = \{\mathbf{v} \in \mathbb{R}^n : v_i \geq 0, i = 1, \ldots, n\}$*. S is a subspace of* $\mathbb{R}^n$*. (Hint: consider* $\mathbb{R}^2$ *first.)*

2.2 *The exercise requires us to prove or disprove the fact that S is a subspace of* $\mathbb{R}^n$*. In order to do so, we firstly have to define what a subspace is and which are its properties:*

According to ***Definition 3.1.****:* $A \subseteq \mathbb{R}^n$ *is a subspace of* $\mathbb{R}^n$ *if and only if* $\forall\ \boldsymbol{x}, \boldsymbol{y} \in A$ *and* $\forall c_1, c_2 \in \mathbb{R}^n$ $c_1\boldsymbol{x} + c_2\boldsymbol{y} \in A$.

In addition, following ***Theorem 3.2.****:* $A \subseteq \mathbb{R}^n$ *is a subspace if and only if all the following three conditions are met:*

- *The zero vector,* ***0*** *belongs to A.*
- *The sum of any two vectors in A belongs to A.*
- *The product of any vector in A by any real number belongs to A.*

S, as defined in the text of the exercise, doesn't satisfy the third condition, hence violating the theorem.

Let's take as an example a vector $\mathbf{v} \in \mathbb{R}^2$*:* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.
To belong to the set S, the condition is that both v_1 *and* v_2 *are* ≥ 0.
Hence, if we enforce the third condition of the ***Theorem 3.2.****, multiplying the vector* $\mathbf{v}$ *by any real number must give back a vector which is still in the subspace.*
However, if we multiply $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \times$ *(-1)* $= \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$

Where $-v_1, -v_2$ *are* ≤ 0*. Hence, the vector does not belong anymore to S, showing that S is not a subspace of* $\mathbb{R}^2$.

For example, let's take $\mathbf{v} = \begin{bmatrix}1\\1\end{bmatrix} \in S$, $\mathbf{v} \times (-1) = \begin{bmatrix}-1\\-1\end{bmatrix} \notin S$.

Therefore, we have found a specific case for which the thesis is not verified. This specific case can be extended to any value of n, because we know, that whichever the number of components of the vector (1, ..., n), when multiplied by a negative real number all its components will change sign and the resulting vector will be outside of S. Hence S is not a subspace of $\mathbb{R}^n$.

Exercise 2.3 (5 points)**.** *Prove or disprove: Let* $S = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| \leq 1\}$. *S is a subspace of* $\mathbb{R}^n$. *(Hint: consider* $\mathbb{R}^2$ *first.)*

2.3 *The exercise requires us to prove or disprove the fact that S is a subspace of* $\mathbb{R}^n$. *In order to do so, we firstly have to define what a subspace is and which are its properties:*

*According to **Definition 3.1.**:* $A \subseteq \mathbb{R}^n$ *is a subspace of* $\mathbb{R}^n$ *if and only if* $\forall\ \boldsymbol{x}, \boldsymbol{y} \in A$ *and* $\forall c_1, c_2 \in \mathbb{R}^n$ $c_1\boldsymbol{x} + c_2\boldsymbol{y} \in A$.

*In addition, following **Theorem 3.2.**:* $A \subseteq \mathbb{R}^n$ *is a subspace if and only if all the following three conditions are met:*

- *The zero vector,* $\boldsymbol{0}$ *belongs to A.*
- *The sum of any two vectors in A belongs to A.*
- *The product of any vector in A by any real number belongs to A.*

S, as defined in the text of the exercise, doesn't satisfy the second and third conditions, hence violating the theorem.

Let's now consider a vector $\in \mathbb{R}^2$ *that belongs to S:*

$$\begin{bmatrix}1\\0\end{bmatrix}$$

It belongs to the set because:

$$\left\| \begin{bmatrix}1\\0\end{bmatrix} \right\| = \sqrt{x^2 + y^2} = 1 \leq 1$$

However, if we try to check the second condition for the subspaces:

$$\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

But,

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin S \rightarrow \left\| \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\| = \sqrt{x^2 + y^2} = \sqrt{4+0} = 2 \geq 1$$

The set S does not even satisfy the third condition, indeed we can simply obtain the same result of the previous check by multiplying:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \times 2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin S \rightarrow \left\| \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\| = \sqrt{x^2 + y^2} = \sqrt{4+0} = 2 \geq 1$$

We have now shown that S is not a subspace of $\mathbb{R}^2$*. Therefore, we have found a specific case for which the thesis is not verified. This specific case can be extended to any value of n, because we know, that whichever the number of components of the vector (1, ..., n), the sum of two or more vectors resulting in a vector with entries* ≥ 1 *will be outside of S. Hence S is not a subspace of* $\mathbb{R}^n$*.*