30407, Mathematics 2 BEMACS
Problem Set 2
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1 Eigenvalues

Exercise 1.1 (14 points). *In the following questions, report only the main steps in the calculations. For point 3, explain in detail your line of reasoning.*

1. *Consider the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x+y \end{bmatrix}.$$

Find the eigenvalues and the corresponding eigenspaces of T.

2. *Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by*

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ x+y+z \\ x+y+z \end{bmatrix}.$$

Find the eigenvalues and the corresponding eigenspaces of T.

3. *Consider now the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_1 + \cdots + x_n & \dots & x_1 + \cdots + x_n \end{bmatrix}.$$

Based on the previous cases, can you find the eigenvalues and the corresponding eigenspaces of T? (Hint: you might start considering what happens in the 4-dimensional case)

1.1 *1. We are searching for the eigenvalues and the corresponding eigenspaces of the linear operator T:*

$$T(\mathbf{x}) = A \cdot \mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To find the eigenvalues we want to evaluate:

$$det(T - \lambda I) = 0$$

$$det(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}) = 0$$

$$(1-\lambda)(1-\lambda) - 1 = 0$$

From which we obtain 2 eigenvalues:

- $\lambda_1 = 0$, *which has algebraic multiplicity = 1*
- $\lambda_2 = 2$, *which has algebraic multiplicity = 1*

Substituing the eigenvalues back in the matrix and solving the associated homogeneous systems:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We obtain:

- $\mathbf{v_1} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, *with* $t \in \mathbb{R}$, *that is the eigenvector relative to* $\lambda_1 = 0$ *(geometric multiplicity = 1). The associated eigenspace has dimension 1 and as a basis* $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.
- $\mathbf{v_2} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, *with* $t \in \mathbb{R}$, *that is the eigenvector relative to* $\lambda_2 = 2$ *(geometric multiplicity = 1). The associated eigenspace has dimension 1 and as a basis* $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

2. *We are searching for the eigenvalues and the corresponding eigenspaces of the linear operator T:*

$$T(\mathbf{x}) = A \cdot \mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To find the eigenvalues we want to evaluate:

$$det(T - \lambda I) = 0$$

$$det(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}) = 0$$

$$(1-\lambda)^3 + 1 + 1 - (1-\lambda) - (1-\lambda) - (1-\lambda) = 0$$

From which we obtain 2 eigenvalues:

- $\lambda_1 = 0$, *which has algebraic multiplicity = 2*
- $\lambda_2 = 3$, *which has algebraic multiplicity = 1*

Substituing the eigenvalues back in the matrix and solving the associated homogeneous systems:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

And

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We obtain:

- $\mathbf{v_1} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{v_2} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$*,with* $t, s \in \mathbb{R}$*, that are the eigenvectors relative to* $\lambda_1 = 0$ *(geometric multiplicity = 2). The associated eigenspace has dimension 2 and as a basis* $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

- $\mathbf{v_3} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *,with* $t \in \mathbb{R}$*, that is the eigenvector relative to* $\lambda_2 = 3$ *(geometric multiplicity = 1). The associated eigenspace has dimension 1 and as a basis* $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

3. *We are now considering the linear operator* T *as going from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$*, as suggested, we will begin analysing the 4-D case:*

$$T(\mathbf{x}) = A \cdot \mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

To find the eigenvalues we want to evaluate:

$$det(T - \lambda I) = 0$$

$$det(\begin{bmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{bmatrix}) = 0$$

$$\lambda^4 - 4\lambda^3 = 0$$

From which we obtain 2 eigenvalues:

- $\lambda_1 = 0$*, which has algebraic multiplicity = 3*
- $\lambda_2 = 4$*, which has algebraic multiplicity = 1*

Substituing the eigenvalues back in the matrix and solving the associated homogeneous systems:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

And

$$\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We obtain:

- $\mathbf{v_1} = t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v_2} = s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{v_3} = p \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, *with* $t, s, p \in \mathbb{R}$ *that are the eigenvectors relative to* $\lambda_1 = 0$ *(geometric multiplicity = 3). The associated eigenspace has dimension 3 and as a basis* $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
- $\mathbf{v_2} = t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ *with* $t \in \mathbb{R}$, *that is the eigenvector relative to* $\lambda_2 = 3$ *(geometric multiplicity = 1). The associated eigenspace has dimension 1 and as a basis* $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Therefore for the general case $T : \mathbb{R}^n \to \mathbb{R}^n$, *we will have:*

$$T(\mathbf{x}) = A \cdot \mathbf{x} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

From which we will obtain 2 eigenvalues:

- $\lambda_1 = 0$, *which has algebraic multiplicity* $= n - 1$
- $\lambda_2 = n$, *which has algebraic multiplicity* $= 1$

Substituing the eigenvalues back in the matrix and solving the associated homogeneous system, we obtain:

- $\mathbf{v_1} = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{v_2} = t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\dots$, $\mathbf{v_{n-1}} = t_{n-1} \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$, *where* $t_1, t_2, ..., t_{n-1} \in \mathbb{R}$

That are the eigenvectors relative to $\lambda_1 = 0$ (geometric multiplicity $= n-1$). The associated eigenspace has dimension $n-1$ and as a basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$.

- $\mathbf{v_n} = t_n \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ *where* $t_n \in \mathbb{R}$

  That is the eigenvector relative to $\lambda_2 = n$ (geometric multiplicity = 1) with n rows. The associated eigenspace has dimension 1 and as a basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$.

2 Differentiable Functions

Exercise 2.1 (8 points). *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ be given by*

$$F(x, y, z) = \begin{bmatrix} 2xy \\ y - z \end{bmatrix}, \quad G(x, y) = \begin{bmatrix} x - 4y \\ xy \\ x + y \end{bmatrix}.$$

Find the differentials $\mathrm{d}F(x, y, z)$, $\mathrm{d}G(x, y)$ and $\mathrm{d}(F \circ G)(x, y)$ directly, and verify the Chain Rule via Matrix Multiplication, that is, verify that

$$\mathrm{d}(F \circ G)(x, y) = \mathrm{d}F(G)\,\mathrm{d}(G)$$

2.1 *According to* ***Definition 4.18*** *the differential of a function can be expressed in form of a matrix, that is, the Jacobian matrix. Hence:*

$$\mathrm{d}F(x, y, z) = \begin{bmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2ex] \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 2y & 2x & 0 \\[2ex] 0 & 1 & -1 \end{bmatrix}$$

$$\mathrm{d}G(x,y) = \begin{bmatrix} \dfrac{\partial G_1}{\partial x} & \dfrac{\partial G_1}{\partial y} \\ \\ \dfrac{\partial G_2}{\partial x} & \dfrac{\partial G_2}{\partial y} \\ \\ \dfrac{\partial G_3}{\partial x} & \dfrac{\partial G_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ \\ y & x \\ \\ 1 & 1 \end{bmatrix}$$

We need now to evaluate the differential of $\mathrm{d}(F \circ G)(x,y)$*, hence:*

$$(F \circ G)(x,y) = F(\begin{bmatrix} x-4y \\ xy \\ x+y \end{bmatrix}) = \begin{bmatrix} 2(x-4y)(xy) \\ (xy)-(x+y) \end{bmatrix} = \begin{bmatrix} 2x^2y - 8y^2x \\ xy - x - y \end{bmatrix}$$

Let's suppose that $H_1(x,y) = 2x^2y - 8xy^2$ *and* $H_2(x,y) = xy - x - y$
Then, the differential expressed as Jacobian matrix is:

$$d(F \circ G)(x,y) = \begin{bmatrix} \dfrac{\partial H_1}{\partial x} & \dfrac{\partial H_1}{\partial y} \\ \\ \dfrac{\partial H_2}{\partial x} & \dfrac{\partial H_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 4xy - 8y^2 & 2x^2 - 16xy \\ \\ y-1 & x-1 \end{bmatrix}$$

Now, we want to verify, via matrix multiplication the Chain Rule, that is:

$$\mathrm{d}(F \circ G)(x,y) = \mathrm{d}F(G)\,\mathrm{d}(G)$$

To do so, we start evaluating F(G):

$$F(G) = F(x-4y,\ xy,\ x+y) = \begin{bmatrix} 2(x-4y)(xy) \\ (xy)-(x+y) \end{bmatrix}$$

Then we take the derivative of F(G) with respect to the components of G, that is:

$$dF(G) = \begin{bmatrix} \frac{\partial F_1}{\partial(x-4y)} & \frac{\partial F_1}{\partial(xy)} & \frac{\partial F_1}{\partial(x+y)} \\ \\ \frac{\partial F_2}{\partial(x-4y)} & \frac{\partial F_2}{\partial(xy)} & \frac{\partial F_2}{\partial(x+y)} \end{bmatrix} = \begin{bmatrix} 2xy & 2x-8y & 0 \\ \\ 0 & 1 & -1 \end{bmatrix}$$

To conclude the demonstration, we calculate:

$$\mathrm{d}F(G)\,\mathrm{d}(G) = \begin{bmatrix} 2xy & 2x-8y & 0 \\ \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ \\ y & x \\ \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4xy - 8y^2 & 2x^2 - 16xy \\ \\ y-1 & x-1 \end{bmatrix}$$

We have thus shown, that the equality $\mathrm{d}(F \circ G)(x,y) = \mathrm{d}F(G)\,\mathrm{d}(G)$ *holds.*

Exercise 2.2 (4 points). *Find the directional derivative of:*

$$f(x,y) = e^{x+y}\ln y$$

at $(0,1)$ *along the direction of* $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

2.2 By definition $\|\mathbf{v}\| = 1$, *so we have to check the norm of the given vector* $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{k} v_i^2} = \sqrt{(-4)^2 + (3)^2} = \sqrt{25} = 5$$

Since $\|\mathbf{v}\| \neq 1$ *we have to normalize the vector:*

$$\mathbf{v}' = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{v}}{5}$$

$$\mathbf{v}' = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

We can evaluate the directional derivative as the gradient calculated in $(0,1)$ *times the direction* $\mathbf{v}$. *This is possibile since* $f(x,y)$ *is differentiable, because its partial derivatives exist and are continuous:*

$$\frac{\partial f}{\partial x} = \ln(y)e^{x+y} \quad \textit{and} \quad \frac{\partial f}{\partial y} = e^{x+y}\ln(y) + \frac{e^{x+y}}{y}$$

Hence, the gradient ∇ *is defined as follows:*

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (\ln(y)e^{x+y}, e^{x+y}\ln(y) + \frac{e^{x+y}}{y})$$

We evaluate now the gradient in the point $(0,1)$*:*

$$\nabla f_{(0,1)} = (0, e)$$

And we finally obtain the ***directional derivative*** *as:*

$$\nabla f_{(0,1)} \cdot \mathbf{v}' = (0,e) \cdot \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} = \frac{3}{5}e$$

Exercise 2.3 (8 points). *Consider a generic direction* $\mathbf{y} = (a,b)$ *(*$\|\mathbf{y}\| = 1$*).*

1. *Find, using the definition of directional derivative, all directions* (a, b) *such that the directional derivative at* $(0, 0)$ *of*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

exists.

2. *Find, using the definition of directional derivative, all directions* (a, b) *such that the directional derivative at* $(0, 0)$ *of*

$$g(x,y) = \begin{cases} \frac{x+y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

exists.

(Hint: use the definition of directional derivative. If you want to use ϕ*, you need to investigate whether* ϕ *is differentiable at* 0 *or not.)*

2.3 *According to **Definition 4.1**: If the limit* $f'(\mathbf{x}, \mathbf{y}) = \lim_{t \to 0^+} \frac{f(\mathbf{x}+t\mathbf{y})-f(\mathbf{x})}{t}$ $\exists$ *and is finite we say the derivative of* f *at* $x \in A$ *along the direction* $\boldsymbol{y}$ *exists and is* $f'(\mathbf{x}, \mathbf{y})$*. So,*

1.

$$f'((0,0),(a,b)) = \lim_{t \to 0^+} \frac{f((0,0)+t(a,b)) - f(0,0)}{t} = \lim_{t \to 0^+} \frac{f(at,bt) - f(0,0)}{t}$$

$$\lim_{t \to 0^+} \frac{\frac{atbt}{a^2t^2+b^2t^2}}{t} = \lim_{t \to 0^+} \frac{\frac{ab}{a^2+b^2}}{t} = \lim_{t \to 0^+} \frac{ab}{t}$$

For this limit to $\exists$*,* $(a \cdot b)$ *has to be equal to 0, and the* $\|y\| = 1$*. Hence the directions are* $(1, 0)$ *and* $(0, 1)$*.*

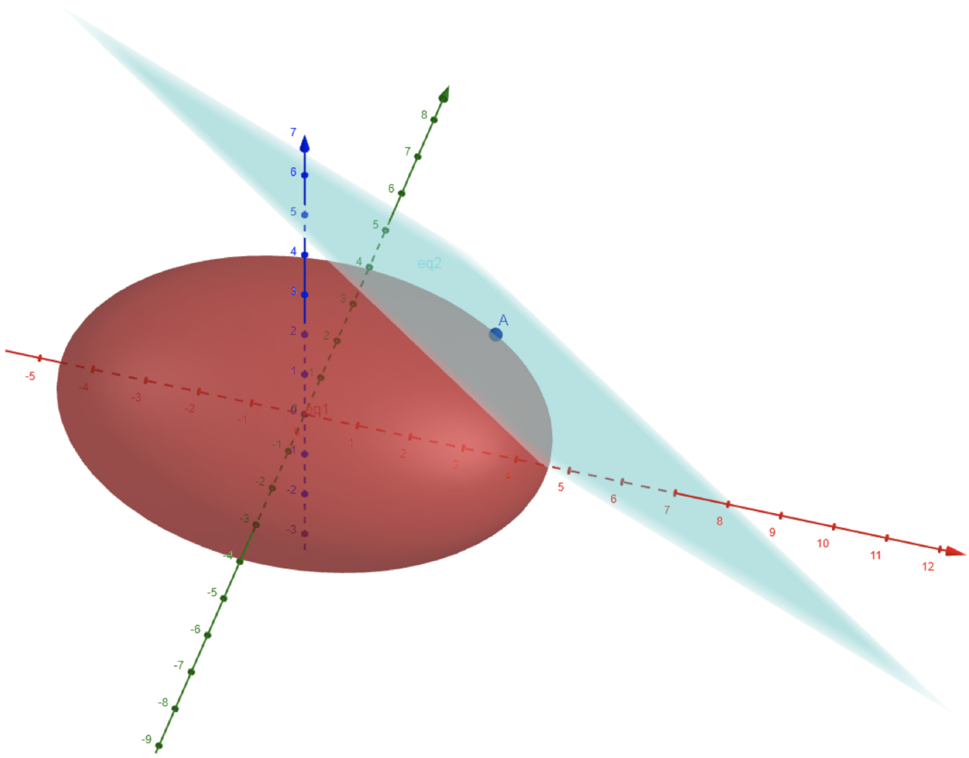
2.

$$f'((0,0),(a,b)) = \lim_{t \to 0^+} \frac{f((0,0)+t(a,b)) - f(0,0)}{t} = \lim_{t \to 0^+} \frac{f(at,bt) - f(0,0)}{t}$$

$$\lim_{t \to 0^+} \frac{\frac{at+bt}{a^2t^2+b^2t^2}}{t} = \lim_{t \to 0^+} \frac{a+b}{t^2(a^2+b^2)} = \lim_{t \to 0^+} \frac{a+b}{t^2}$$

For this limit to $\exists$*, (a+b) has to be equal to 0, and the* $\|y\| = 1$*. Hence the directions are* $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ *and* $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$*.*

Exercise 2.4 (6 points)**.** *Let E be the ellipsoid with equation $x^2+2y^2+4z^2 = 21$. Find an equation of the plane tangent to E at $(3,2,1)$. (see figure below.)*



2.4 *Since we want to evaluate the equation of the plane tangent to the ellipsoid in the point $(3,2,1)$, we need the vector perpendicular to the surface in the point $(3,2,1)$. Hence, we start by evaluating the gradient:*

$$\nabla E = \left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}, \frac{\partial E}{\partial z}\right) = (2x, 4y, 8z)$$

We evaluate the gradient in the point $(3,2,1)$, which represents our normal vector to define the plane:

$$\nabla E_{(3,2,1)} = (6, 8, 8)$$

We are now able to calculate the equation of the plane starting from the generic one:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Where with (a, b, c) *we refer to the components of the normal vector and* x_0, y_0 *and* z_0 *are the coordinates of the point* $(3, 2, 1)$. *Hence:*

$$6(x-3)+8(y-2)+8(z-1)=0$$

After some simplifications, we can reach the equation of the plane in the following form:

$$6x+8y+8z=42$$

$$3x+4y+4z=21$$