30407, Mathematics 2 BEMACS
Problem Set 3
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1 The Hessian Matrix

Exercise 1.1 (12 points). *If $f : \mathbb{R}^2 \to \mathbb{R}$ is twice differentiable and $H_f(x, y)$ is the Hessian matrix at (x, y), then the definiteness of the quadratic form can be found using the determinant of H_f as in theorem 5.9 in the notes. On the other side, the definiteness of H_f is defined through the eigenvalues of H_f. Can you formally justify the link between the two ideas?*

1.1 *According to **Definition 3.74**: A quadratic form on $\mathbb{R}^n$ is a real-valued function of the form $Q(\mathbf{x}) = \sum\limits_{i \leq j} a_{i,j} x_i x_j$ in which each term is a monomial of degree two. Each quadratic form Q can be represented by a symmetric matrix A so that: $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.*

Considering $f : \mathbb{R}^2 \to \mathbb{R}$ and applying the above mentioned definition, we can rewrite f as:

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

*where A is a symmetric matrix. We can take P as the matrix of which columns are eigenvectors (This can be done by the **Spectral Theorem** and Eigendecomposition). Especially, P has orthonormal column vectors.*

Hence,

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T (P^T AP)\mathbf{y}$$

Where $(P^T AP)$ is a diagonal matrix of which diagonal entries are the eigenvalues of A, and $\mathbf{y} = P^T\mathbf{x}$.

Such Quadratic form can be written as:

$$f(\mathbf{x}) = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

This new expression for f is beneficial to characterize the definitness of the form because we no longer have a "mixed" term of the form $y_1 y_2$. Since we know that y_1^2 and y_2^2 will both be non-negative for all vectors $\mathbf{x}$, we are able to characterize the definiteness of f from the signs of the eigenvalues λ_1 and λ_2.

We want now to check the relation between the definiteness of the quadratic form found using the determinant of H_f and the definiteness of H_f found through the eigenvalues of H_f.

So, on the other side, we define H_f:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial y_1^2} & \frac{\partial^2 f}{\partial y_1 \partial y_2} \\ \\ \frac{\partial^2 f}{\partial y_1 \partial y_2} & \frac{\partial^2 f}{\partial y_2^2} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ \\ 0 & \lambda_2 \end{bmatrix}$$

$$det(H_f) = \lambda_1 \lambda_2$$

We can now clearly notice the relation between the determinant of the Hessian and the definiteness of the quadratic form, which passes through the eigenvalues. Indeed:

- *if $\lambda_1, \lambda_2 > 0$, f is positive definite*
- *if $\lambda_1, \lambda_2 < 0$, f is negative definite*
- *if $\lambda_1 > 0$ and $\lambda_2 = 0$, f is positive semi-definite*
- *if $\lambda_1 = 0$ and $\lambda_2 < 0$, f is negative semi-definite*
- *if $\lambda_1 > 0$ and $\lambda_2 < 0$, f is indefinite*

2 Optimization

Exercise 2.1 (14 points)**.** *Find the maximum and minimum values of* $f(x,y) = 81x^2 + y^2$ *subject to the constraint* $4x^2 + y^2 = 9$.

2.1 *First of all, we recognize that the constraint is a compact set. This tells us that, if the function is continuous on the set, according to **Extreme Value Theorem**, f must attain its max and min on the set. This is our case, because the function is continuous on all its domain* $\mathbb{R}^2$*, hence also on the compact set defined by the constraint.*

*We proceed now in evaluating the **Lagrangian function**:*

$$L(x,y,\lambda) = 81x^2 + y^2 + \lambda(9 - 4x^2 - y^2)$$

Hence, we take the partial derivatives:

$$\frac{\partial L}{\partial x} = 162x - 8\lambda x; \quad \frac{\partial L}{\partial y} = 2y - 2\lambda y; \quad \frac{\partial L}{\partial \lambda} = 9 - 4x^2 - y^2$$

Setting them equal to 0, allow us to identify the stationary points:

$$\begin{cases} 162x - 8\lambda x = 0 \\ 2y - 2\lambda y = 0 \\ 9 - 4x^2 - y^2 = 0 \end{cases} \qquad \begin{cases} 2x(81 - 4\lambda) = 0 \\ 2y(1 - 2\lambda) = 0 \\ 9 - 4x^2 - y^2 = 0 \end{cases}$$

From which we obtain:

$$\begin{cases} x = \frac{3}{2} \\ y = 0 \\ \lambda = \frac{81}{4} \end{cases} \quad \begin{cases} x = -\frac{3}{2} \\ y = 0 \\ \lambda = \frac{81}{4} \end{cases} \quad \begin{cases} x = 0 \\ y = 3 \\ \lambda = 1 \end{cases} \quad \begin{cases} x = 0 \\ y = -3 \\ \lambda = 1 \end{cases}$$

Since f(x,y) is continuous on the compact set defined by the constraint $4x^2 + y^2 = 9$*, we know that* $\exists$ *Max and Min. So, to find them, we make a comparison between the values of the function at the stationary points:*

$$f(\frac{3}{2}, 0) = \frac{729}{4} \quad \textit{and} \quad f(-\frac{3}{2}, 0) = \frac{729}{4}$$
$$f(0,3) = 9 \quad \textit{and} \quad f(0,-3) = 9$$

From the comparison we can clearly notice that:

- $(\frac{3}{2}, 0)$ *and* $(-\frac{3}{2}, 0)$ *are **maximizers** and* $\frac{729}{4}$ *is the **maximum**.*

- $(0, 3)$ *and* $(0, -3)$ *are* ***minimizers*** *and 9 is the* ***minimum****.*

Exercise 2.2 (14 points)**.** *Let* $f(x, y) = 3x^4 - 4x^2y + y^2$*. Show that on every line* $y = mx$ *the function has a minimum at* $(0, 0)$*, but that there is no relative minimum in any two-dimensional neighborhood of the origin. (Hint: make a sketch indicating the set of points* (x, y) *at which* $f(x, y) > 0$ *and the set at which* $f(x, y) < 0$*.)*

2.2 *We will start by showing that the function has a minimum at* $(0, 0)$ *on every line* $y = mx$*.*
To do so, we substitute $y = mx$ *in the equation:*

$$f(x, y) = 3x^4 - 4x^2y + y^2; \quad f(x, mx) = 3x^4 - 4x^2(mx) + (mx)^2 = 3x^4 - 4x^3m + m^2x^2$$

We start by evaluating the ***First-Order condition****:*

$$\frac{\partial f(x, mx)}{\partial x} = 0$$

$$12x^3 - 12x^2m + 2xm^2 = 0$$

This condition returns us three stationary points:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} ; \quad \begin{cases} x = \frac{m}{6}(3 - \sqrt{3}) \\ y = \frac{m^2}{6}(3 - \sqrt{3}) \end{cases} ; \quad \begin{cases} x = \frac{m}{6}(3 + \sqrt{3}) \\ y = \frac{m^2}{6}(3 + \sqrt{3}) \end{cases}$$

Since the exercise requires us to analyse the point (0,0) and its neighborhood, we will ignore the other two points from now on, and we will discuss the ***Second-Order Condition*** *in the origin.*

$$\frac{\partial^2 f(x, mx)}{\partial x^2} = 36x^2 - 24xm + 2m^2$$

We want to evaluate now the second derivative at (0,0):

$$36(0)^2 - 24(0)m + 2m^2 = 2m^2$$

From which we can easily notice that the second derivative is always positive and, hence, (0,0) is a ***minimizer*** *on every line* $y = mx$*. However, to show that there is no relative minimum in any two-dimensional neighborhood of the origin, we will use the constrained optimization view of the problem, exploiting the Lagrangian function and the Hessian Matrix:*

$$L(x, y, \lambda) = 3x^4 - 4x^2y + y^2 + \lambda(mx - y)$$

Then, we find the stationary points of the Lagrangian:

$$\frac{\partial L}{\partial x} = 12x^3 - 8xy + \lambda m; \quad \frac{\partial L}{\partial y} = -4x^2y + y - \lambda; \quad \frac{\partial L}{\partial \lambda} = mx - y$$

$$\begin{cases} 12x^3 - 8xy + \lambda m = 0 \\ -4x^2y + y - \lambda = 0 \\ mx - y = 0 \end{cases}$$

From which we obtain, the same coordinate we found earlier.
As before, we are only interested in what happens in the origin (x = 0, y = 0, $\lambda = 0$), so, we evaluate the Hessian at (0,0,0):

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 36x^2 - 8y & -8x \\ \\ -8x & 2 \end{bmatrix}$$

$$H_{f(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$det(H_{f(0,0)}) = 0$, so we need to discuss the sign of the function in the neighborhood of the origin:

$$f(x, y) - f(0, 0) = 3x^4 - 4x^2y + y^2$$

If, unlike what we have done before, we approach the origin from a parabola $y = mx^2$, we can express the function in the following way:

$$f(x, mx^2) - f(0, 0) = 3x^4 - 4mx^4 + m^2x^4$$

Studying the sign of it:

$$3x^4 - 4mx^4 + m^2x^4 > 0$$

We can see that the function is positive for every value of x different from 0, but only if $m < 1$ or $m > 3$, which means that $(0, 0)$ cannot be a minimum, but rather a saddle point.