30407, Mathematics 2 BEMACS
Problem Set 4
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1 Implicit Function Theorem

Exercise 1.1 (20 points). *Suppose $F(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is defined by $F(x, y) = xe^y + ye^x$.*

1. *Verify that there exist a neighborhood of $x_0 = 0$ where the IFT hypotheses are satisfied and thus a unique continuously differentiable function $y = y(x)$ exists such that $y(0) = 0$ and $F(x, y(x)) = 0$.*
2. *Find $y'(0)$*
3. *Find $y''(0)$*

1.1 *The Implicit Function Theorem states that. Let $A \subseteq \mathbb{R}^n, n \geq 1$, and $B = (b_1, b_2), -\infty \leq b1 \leq b2 \leq +\infty$ an interval in $\mathbb{R}$. Let $f : A \times B \to \mathbb{R}$ and such that the following hypotheses are satisifed:*

- *for every $\mathbf{x} \in A, f(\mathbf{x}, \cdot)$ is continuous and strictly increasing in B:*
- *for every $\mathbf{x} \in A$ we have:*

$$\lim_{x \to b_1^+} f(\mathbf{x}, y) < 0 \quad \text{and} \quad \lim_{x \to b_2^-} f(\mathbf{x}, y) > 0$$

Then there exists one and only one function $y = y(\mathbf{x}) : A \to B$ such that:

$$f(\mathbf{x}, y(\mathbf{x})) = 0 \quad \forall \mathbf{x} \in A$$

1. *We want indeed to show that the function $F(x, y) = xe^y + ye^x$ satisfies the above hypotheses in a neighborhood of $x_0 = 0$ and thus a unique continuously differentiable function $y = y(x)$ exists such that $y(0) = 0$ and $F(x, y(x)) = 0$. So:*

- *The function $F(x,y) = xe^y + ye^x$ is always continuous and*

$$\frac{\partial F}{\partial y} = xe^y + e^x$$

Furthermore, since we are discussing a neighborhood of $x_0 = 0$, we can substitute that value in the equation and we obtain:

$$\frac{\partial F}{\partial y} = 1 > 0$$

From which we can see that the function is strictly increasing in a neighborhood of $x_0 = 0$.

- *We want now to analyze the limits of the function in a neighborhood of $x_0 = 0$, as y tends to the extremes of its domain:*

$$\lim_{y \to -\infty} F(0,y) = -\infty \quad \textit{and} \quad \lim_{y \to +\infty} F(0,y) = +\infty$$

We have shown that the two hypotheses of the Implicit Function Theorem are satisfied and thus a unique continuously differentiable function $y = y(x)$ exists. Since $F(x,y)$ passes through the origin, the conditions: $y(0) = 0$ and $F(x, y(x)) = 0$ are verified.

2. *We are now searching $y'(0)$.*
   To do so we derive $F(x, y(x))$ with respect to x, and we set it equal to 0:

$$\frac{\partial F(x,y(x))}{\partial x} = 0$$
$$e^y + y'xe^y + ye^x + ye^x = 0$$

   We solve for y':

$$y' = -\frac{ye^x + e^y}{xe^y + e^x}$$

   From which we can derive, setting $x = 0$ and knowing that $y(0) = 0$, that:

$$y'(0) = -1$$

3. *The last thing we want to evaluate is $y''(0)$.*
   To do so we derive $\frac{\partial F(x,y(x))}{\partial x}$ with respect to x, and we set it equal to 0:

$$\frac{\partial^2 F(x,y(x))}{\partial x^2} = 0$$

$$y'e^y + y''xe^y + y'e^y + x(y')^2e^y + y''e^x + y'e^x + y'e^x + e^xy = 0$$

Setting x=0 and knowing that y(0) = 0 and $y'(0) = -1$. *We solve for* y'' *and we get that:*

$$y''(0) = 4$$

2 Double Integrals

Exercise 2.1 (20 points)**.** *Find*

$$\iint_R (x+y)\,\mathrm{d}A$$

where R *is the region bounded by* $y = x$, $y = 3x$ *and* $x + y = 4$. *Use the transformation* $x = u + v$, $y = u - v$.

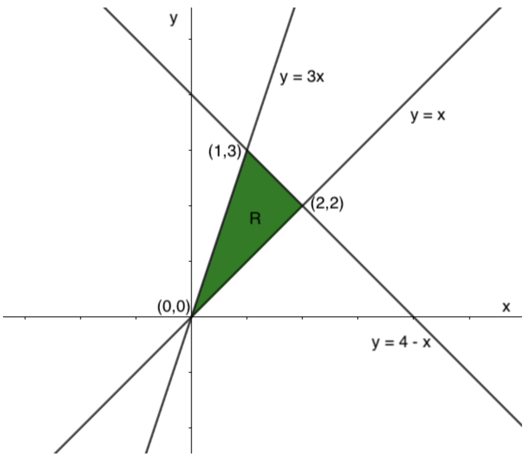
2.1 *We are going to evaluate the intersections among the three different lines to better understand the nature of the region* $R = \{(x,y) : y = x \wedge y = 3x \wedge x + y = 4\}$*:*

$$\begin{cases} y = x \\ y = 4 - x \end{cases} = \begin{cases} y = 2 \\ x = 2 \end{cases} \quad \textit{From which we obtain } (2,2)$$

$$\begin{cases} y = 3x \\ y = 4 - x \end{cases} = \begin{cases} y = 3 \\ x = 1 \end{cases} \quad \textit{From which we obtain } (1,3)$$

$$\begin{cases} y = 3x \\ y = x \end{cases} = \begin{cases} y = 0 \\ x = 0 \end{cases} \quad \textit{From which we obtain } (0,0)$$

Hence, the representation of the region R is the following:



Where the green area can be found as:

$$\begin{cases} y > x \\ y < 3x \\ y < 4 - x \end{cases}$$

Now, we need to transform the xy-region R into the uv-region R'. To do so, we need to express (u,v) in terms of (x,y).

$$\begin{cases} x = u + v \\ y = u - v \end{cases}$$

$$u = \frac{x+y}{2} \qquad \textit{and} \qquad v = \frac{x-y}{2}$$

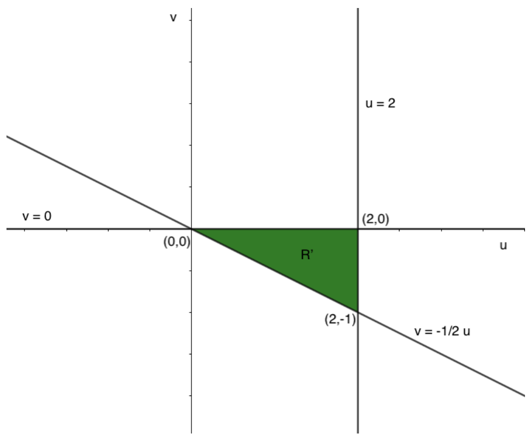
Then, we have the following correspondence ($xy \to uv$):

$$(0,0) \to (0,0),\ (2,2) \to (2,0),\ (1,3) \to (2,-1)$$

We need to transform also the system to define the green area as follows:

$$\begin{cases} y > x \\ y < 3x \\ y < 4 - x \end{cases} \to \begin{cases} u - v < u + v \\ u - v < 3(u+v) \\ u - v < 4 - (u+v) \end{cases} = \begin{cases} v < 0 \\ v > -\frac{1}{2}u \\ u < 2 \end{cases}$$

And the we have that the region R' is a triangle in the uv-plane.



From the same equations used before, we have:

$$x + y = u + v + u - v = 2u$$

According to ***Theorem 5.7*** *of the Notes on Double Integrals, we have to evaluate the Jacobian Matrix and its determinant. Since the transformation is linear, the Jacobian is a constant matrix:*

$$J = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$det(J) = -1 - (1) = -2$$

From the previous system we define the extremes of integration of u and v, respectively as: $(0, 2)$ *and* $(0, -\frac{1}{2}u)$.

Finally, we can evaluate the double integral:

$$\iint_R (x+y)\,\mathrm{d}A = \iint_{\mathbb{R}'} 2u|-2|\ dvdu = \int_0^2 \int_{-\frac{1}{2}u}^0 4u\ dvdu = \int_0^2 2u^2\ du = \frac{16}{3}$$