Mathematics (BEMACS) - A. Y. 2019/20

Exercises - Set 1

1.1 Let's consider the sets $A = (-\infty, 5]$ and $B = (-25, +\infty)$, where $\mathbb{R} = (-\infty, +\infty)$ is the universal set. Find the sets indicated below:

(a) $A \cup B$ (b) $A \cap B$ (c) $A^c = \mathbb{R} \setminus A$

(d) $B^c = \mathbb{R} \setminus B$ (e) $\mathbb{R} \setminus (A \cap B)$ (f) $(\mathbb{R} \setminus A)^c \cap B^c$

(g) Show that $\mathbb{R} \setminus (A \cap B) = (\mathbb{R} \cap A^c) \cup (\mathbb{R} \cap B^c)$.

1.2 Consider the following subsets of $\mathbb{R}$ (the set of real numbers)

$$A = \left\{x \in \mathbb{R} : x^2 < 1 \text{ e } x > 0\right\} \quad B = \left\{x \in \mathbb{R} : 3^{x^2-1} \leq 27\right\}$$

(a) Does any relationship exist between A and B? And between A^c and B^c?

(b) Calculate the following sets $A \cap B$, $A \cup B$, $\mathbb{R} \setminus (A \cap B)$ and $\mathbb{R} \setminus (A^c \cap B)$.

1.3 Let x be a real number. (a) Prove that $|x|^2 = x^2$, for any x; (b) Show that

$$\frac{x^2 - 1}{|x| + 1} = |x| - 1 \quad ; \quad \left(1 - \frac{|x|}{x}\right)(x + |x|) = 0 \text{ (with } x \neq 0)$$

and $(|2x| - 1)^2 - (1 - |-4x|) = 4x^2$.

1.4 Let A, B be subsets of E (A, $B \subseteq E$). Show that

$$A \setminus B = A \cap B^c$$

that is, the difference between A and B is equal to A intersection B^c (the complement of B with respect to E).

1.5 (a) Let x be a real number. Show that

$$\min\left[4 - \max\left(|x|, 2\right), 0\right] = \begin{cases} 4 - |x| & |x| \geq 4 \\ 0 & |x| < 4 \end{cases}$$

(b) Write the neighborhood of the point $x_0 = 3$ with radius 2 and the neighborhood of the point $x_0 = -2$ with radius $3/2$.

1.6 Let a, b be real non-negative numbers. Show that if $a \leq b$ then

(a) $\sqrt{a} \leq \sqrt{b}$ (b) $a \leq (a + b)/2 \leq b$ (c) $a \leq \sqrt{ab} \leq b$ (d) $\sqrt{ab} \leq (a + b)/2$

and establish when, in each case, the equality sign holds.

1.1 (a) $\mathbb{R}$; (b) $(-25, 5]$; (c) $A^c = \mathbb{R} \setminus A = (5, +\infty)$; (d) $B^c = \mathbb{R} \setminus B = (-\infty, -25]$; (e) we have

$$\mathbb{R} \setminus (A \cap B) = \mathbb{R} \setminus (-25, 5] = (-\infty, -25] \cup (5, +\infty)$$

(e) We have

$$(\mathbb{R} \setminus A)^c \cap B^c = (A^c)^c \cap B^c = A \cap B^c = (-\infty, -25]$$

1.2 (a) $A \subset B$ ("A is a proper subset of B" or "A is strictly contained in B"). $B^c \subset A^c$; (b) $A = (0, 1)$, $B = [-2, 2]$ and $A \cap B = (0, 1)$. We get

$$\mathbb{R} \setminus (A \cap B) = (-\infty, 0] \cup [1, +\infty) \qquad \text{and} \qquad \mathbb{R} \setminus (A^c \cap B) = (-\infty, -2) \cup (0, 1) \cup (2, +\infty)$$

1.3 (a) Use the definition of absolute value; (b) For any $x \in \mathbb{R} \neq 0$ (knowing that $|x|^2 = x^2$) we have

$$\left(1 - \frac{|x|}{x}\right)(x + |x|) = x - |x| + |x| - \frac{|x|^2}{x} = x - x = 0$$

1.4 To establish the equality between the two sets $A \setminus B$ and $A \cap B^c$ we need to consider two cases: (1) $A \setminus B \subseteq A \cap B^c$; (2) $A \cap B^c \subseteq A \setminus B$.

1.5 (a) Use the definition of maximum and minimum between two numbers. Starting from $\max(|x|, 2)$, we must consider two cases:

- if $|x| \geq 2$, then $\max(|x|, 2) = |x|$ and

$$\min[4 - |x|, 0] = \begin{cases} 4 - |x| & \text{if} \quad |x| \geq 4 \\ 0 & \text{if} \quad 2 \leq |x| < 4 \end{cases}$$

- if $|x| < 2$, then $\max(|x|, 2) = 2$ and

$$\min[4 - 2, 0] = 0$$

Therefore the formula holds for any x real number. (b) We have $B_2(3) = \{x \in \mathbb{R} : |x - 3| < 2\} = (1, 5)$ and $B_{3/2}(-2) = \{x \in \mathbb{R} : |x + 2| < 3/2\} = (-7/2, -1/2)$.

1.6 **(a)** We know that $a = (\sqrt{a})^2$ and $b = \left(\sqrt{b}\right)^2$, with a, b non-negative real numbers. We have

$$a \leq b \;\Rightarrow\; (\sqrt{a})^2 \leq \left(\sqrt{b}\right)^2 \;\Rightarrow\; \sqrt{a} \leq \sqrt{b}$$

and the converse is also true, therefore $a \leq b \Leftrightarrow \sqrt{a} \leq \sqrt{b}$ (where the equality sign holds if and only if $a = b$); **(b)** Use the real number properties; **(c)** Use the real number properties; **(d)** We want to show that the arithmetic mean of two numbers is less than or equal to their geometric mean. We have

$$\frac{a + b}{2} \geq \sqrt{ab} \quad \Rightarrow \quad (a + b)^2 \geq 4ab \quad \Rightarrow \quad (a - b)^2 \geq 0$$

true for any a, $b \geq 0$ (where the equality sign holds if and only if $a = b$).