Mathematics (BEMACS) - A. Y. 2019/20

Exercises - Set 2

1.1 Let x be a real number. Solve the inequalities with absolute values:

(a) $(|x|-1)^2 \le 4$ (b) $(|2x|-1)^2 \ge |-4|$ (c) $(|2x|-1)^2 \le 4\,(1-|-x|)^2$

1.2 In $\mathbb{R}^3$, consider the following vectors $\mathbf{x}^1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{x}^2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{x}^3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(a) Find the vector $\mathbf{z} = 2\mathbf{x}^1 - 3\mathbf{x}^2 - \mathbf{x}^3$.

(b) Calculate the norm of $\mathbf{x}^1$ and the norm of the difference between $\mathbf{x}^2$ and $\mathbf{x}^3$ (that is $\left\|\mathbf{x}^2 - \mathbf{x}^3\right\|$).

(c) Show that

$$\left\|\mathbf{x}^1 - \mathbf{x}^2\right\| < \left\|\mathbf{x}^1 - \mathbf{x}^3\right\| + \left\|\mathbf{x}^2 - \mathbf{x}^3\right\|$$

1.3 Consider the following vectors on $\mathbb{R}^3$

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$$

(a) Calculate $(\mathbf{x}, \mathbf{y})$, $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{z}, \mathbf{x})$.

(b) Show that $|(\mathbf{x}, \mathbf{y})| \le \|\mathbf{x}\| \, \|\mathbf{y}\|$.

(c) Calculate $(2\mathbf{x}, 3\mathbf{y})$, $(2\mathbf{x} + 3\mathbf{y}, \mathbf{z})$ and $(2\mathbf{x} - 3\mathbf{z}, 2\mathbf{y})$.

1.4 Let $\mathbf{x}^1$, $\mathbf{x}^2$ be vectors in $\mathbb{R}^3$. Find, if they exist, α such that the following vectors

$$\mathbf{x}^1 = \begin{bmatrix} -\alpha \\ 2 \\ \alpha \end{bmatrix} \qquad \mathbf{x}^2 = \begin{bmatrix} \alpha \\ -1 \\ 3 \end{bmatrix}$$

are orthogonal (or perpendicular) in $\mathbb{R}^3$.

1.5 Let $\mathbf{x}$, $\mathbf{y}$ be vectors in $\mathbb{R}^n$. Show that

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\,\|\mathbf{x}\|^2 + 2\,\|\mathbf{y}\|^2$$

for any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$.

1.6 Let $n \ge 1$ be a natural number. Using the principle of mathematical induction, show that

$$\left\|x^1 + x^2 + \cdots + x^n\right\| \le \left\|x^1\right\| + \left\|x^2\right\| + \ \cdots + \|x^n\|$$

for any $x^1, x^2, \cdots, x^n \in \mathbb{R}^n$.

1.1 (a) $-3 \le x \le 3$; **(b)** From $(|2x|-1)^2 \ge |-4|$ (where $|-4| = 4$) we get

$$|2x|-1 \le -2 \quad \text{or} \quad |2x|-1 \ge 2 \quad \Rightarrow \quad |2x| \le -1 \quad \text{or} \quad |2x| \ge 3 \quad \Rightarrow \quad |x| \ge 3/2$$

therefore $x \le -3/2$ or $x \ge 3/2$; **(c)** $-3/4 \le x \le 3/4$.

1.2 (a) We have

$$2\mathbf{x}^1 - 3\mathbf{x}^2 - \mathbf{x}^3 = \begin{bmatrix} -9 \\ 6 \\ 7 \end{bmatrix}$$

(b) $\|\mathbf{x}^1\| = \sqrt{14}$, $\|x^2 - x^3\| = \sqrt{6}$; **(c)** $\sqrt{27} < \sqrt{21} + \sqrt{6}$.

1.3 (a) From the definition of scalar product between two vectors, we have

$$(\mathbf{x}, \mathbf{y}) = \sum_{s=1}^{3} x_s y_s = (-1)\cdot 2 + 2\cdot(-1) + 2\cdot 3 = 2$$

in the same way we get $(\mathbf{y}, \mathbf{z}) = 2$ and $(\mathbf{z}, \mathbf{x}) = -11$; **(b)** From the norms of $\mathbf{x}$ and $\mathbf{y}$

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})} = 3 \quad \text{and} \quad \|\mathbf{y}\| = \sqrt{(\mathbf{y}, \mathbf{y})} = \sqrt{14}$$

we have

$$|(\mathbf{x}, \mathbf{y})| = 2 < 3\sqrt{14} = \|\mathbf{x}\| \, \|\mathbf{y}\|$$

(c) $(2\mathbf{x}, 3\mathbf{y}) = 12$, $(2\mathbf{x} + 3\mathbf{y}, \mathbf{z}) = -16$ and

$$(2\mathbf{x} - 3\mathbf{z}, 2\mathbf{y}) = (2\mathbf{x}, 2\mathbf{y}) + (-3\mathbf{z}, 2\mathbf{y}) = 4\,(\mathbf{x}, \mathbf{y}) - 6\,(\mathbf{z}, \mathbf{y}) = -4$$

1.4 From the orthogonality condition between two vectors $(\mathbf{x}^1, \mathbf{x}^2) = 0$ (i.e. the scalar product must be null) we have $\alpha^2 - 3\alpha + 2 = 0$, with solutions $\alpha = 1$ and $\alpha = 2$.

1.5 Remember that $\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y})$, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We have

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}) = \\
&= (\mathbf{x}, \mathbf{x} + \mathbf{y}) + (\mathbf{y}, \mathbf{x} + \mathbf{y}) + (\mathbf{x}, \mathbf{x} - \mathbf{y}) - (\mathbf{y}, \mathbf{x} - \mathbf{y}) = \\
&= (\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}) + (\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}) = \\
&= 2\,(\mathbf{x}, \mathbf{x}) + 2\,(\mathbf{y}, \mathbf{y}) = 2\,\|\mathbf{x}\|^2 + 2\,\|\mathbf{y}\|^2
\end{aligned}$$

because we know that $(\mathbf{x}, \mathbf{x}) = \|\mathbf{x}\|^2$, for any $\mathbf{x} \in \mathbb{R}^n$.

1.6 The proposition $P(n)$ is given by

$$\|x^1 + x^2 + \cdots + x^n\| \le \|x^1\| + \|x^2\| + \cdots + \|x^n\|$$

while the proposition $P(n+1)$ is given by

$$\|x^1 + x^2 + \cdots + x^n + x^{n+1}\| \le \|x^1\| + \|x^2\| + \cdots + \|x^n\| + \|x^{n+1}\|$$