Mathematics (BEMACS) - A. Y. 2019/20

Exercises - Set 3

1.1 Consider on $\mathbb{R}^2$ the following vectors

$$\mathbf{x}^1 = \begin{bmatrix} \alpha + 1 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^2 = \begin{bmatrix} 3 \\ \alpha - 2 \end{bmatrix}$$

(a) Find, if they exist, real numbers α such that $\left\|\mathbf{x}^1\right\| = \left\|\left(-\sqrt{2}\right)\mathbf{x}^2\right\|$.

(b) Find, if they exist, real numbers α such that $\left\|\mathbf{x}^1 - \mathbf{x}^2\right\| = 4$

1.2 Let A be a subset of $\mathbb{R}$ defined by $A = (-\infty, 5)$.

(a) Does A have an upper bound? If so, find A^*.

(b) Does A have a lower bound? If so, find A_*.

(c) Consider $B = A^c$. Find B^* and B_*.

1.3 Let A be a subset of $\mathbb{R}$ defined by

$$A = \{x \in \mathbb{R} : |-x| + |x| < 2\}$$

(a) Does A have an upper bound? A lower bound?

(b) Find A^* and A_*.

(c) Let B be the complement of A (with respect to $\mathbb{R}$). Find B^* and B_*.

1.4 Consider the following subsets of $\mathbb{R}$

$$A = (-\infty, 2) \qquad B = [5, +\infty) \qquad C = (-10, 10]$$

(a) Are A, B, C bounded?

(b) Are A^c, B^c, C^c bounded?

(c) Are $A \cup B$, $A \cap C$ and C^c bounded?

1.5 Let n be a natural number. Consider the following subsets of real numbers

$$A = \left\{x \in \mathbb{R} : x = n^2 - 1\right\} \quad \text{and} \quad B = \left\{x \in \mathbb{R} : x = \frac{1}{n+1} - 1\right\}$$

(a) Are A and B bounded or unbounded?

(b) Find, if they exist, the infimum, supremum, maximum and minimum of A and B.

1.6 Let n be a natural number. Consider the following subsets of real numbers

$$A = \{x \in \mathbb{R} : x = e^{-n} - 1\} \quad \text{and} \quad B = \left\{x \in \mathbb{R} : x = \sqrt{\frac{4}{n+1}}\right\}$$

(a) Are A and B bounded or unbounded?

(b) Find, if they exist, the infimum, supremum, maximum and minimum of A and B.

1.1 **(a)** From $\|\mathbf{x}^1\| = \|(-\sqrt{2})\,\mathbf{x}^2\|$ we have $\|\mathbf{x}^1\|^2 = 2\,\|\mathbf{x}^2\|^2$, therefore by substitution we get

$$(\alpha+1)^2 + 16 = 9 + (\alpha-2)^2 \quad \Rightarrow \quad \alpha^2 - 10\alpha + 9 = 0$$

therefore we obtain $\alpha = 1$ or $\alpha = 9$. **(b)** $\alpha = 2$ or $\alpha = 6$.

1.2 **(a)** Yes, A has at least an upper bound. $A^* = [5, +\infty)$; **(b)** No, A does not have a lower bound. $A_* = \emptyset$; **(c)** $B = [5, +\infty)$, $B^* = \emptyset$ (no upper bounds), $B_* = (-\infty, 5]$ (B has at least a lower bound).

1.3 $A = (-1, 1)$. **(a)** A has an upper and a lower bounds; **(b)** $A^* = [1, +\infty)$, $A_* = (-\infty, -1]$; **(c)** We have

$$B = A^c = \mathbb{R} \setminus A = (-\infty - 1) \cup (1, +\infty)$$

and $B^* = B_* = \emptyset$.

1.4 **(a)** A, B unbounded and C bounded; **(b)** A^c, B^c and C^c unbounded; **(c)** $A \cup B$ unbounded, $A \cap C$ bounded and C^c unbounded.

1.5 **(a)** We have $n^2 \geq 0 \Rightarrow n^2 - 1 \geq -1$, therefore any element of A will in the interval $[-1, +\infty)$. We observe that

- $A_* = (-\infty, -1]$ (A has a lower bound);
- $A^* = \emptyset$ (A does not have an upper bound).

A is not bounded from above in $\mathbb{R}$. B is bounded in $\mathbb{R}$; **(b)** $\inf(A) = -1 \in A$ (therefore -1 is the minimum of A), $\sup(A) = +\infty$, $\inf(B) = -1 \notin B$, $\sup(B) = 0 \in B$ (therefore 0 is the maximum of B).

1.6 We have

$$A = \left\{0, \frac{1}{e} - 1, \frac{1}{e^2} - 1, ...\right\} \qquad \text{and} \qquad B = \left\{2, \sqrt{2}, \sqrt{\frac{4}{3}}, 1, \sqrt{\frac{4}{5}}, ...\right\}$$

(a) A is bounded in $\mathbb{R}$, B is bounded in $\mathbb{R}$; **(b)** $\inf(A) = -1 \notin A$ (therefore -1 is not the minimum of A), $\sup(A) = \max A = 0$, $\inf(B) = 0 \notin B$ (therefore 0 is not the minimum of B), $\sup(B) = 2 \in B$ (therefore 2 is the maximum of B).