Mathematics (BEMACS) - A. Y. 2019/20

Exercises - Set 4

1.1 Let x be a non-negative real number such that $x < \varepsilon$ for any $\varepsilon > 0$. Show that $x = 0$.

1.2 Let n be a natural number and let a, b, c be real numbers with $a < b < c$. Consider the following subsets of real numbers

$$A = \{x \in \mathbb{R} : x = (x - a)(x - b)(x - c) < 0\} \quad \text{and} \quad B = \{x \in \mathbb{R} : x = 1 + (-1)^n\}$$

and

$$C = \left\{ x \in \mathbb{R} : x = 1 + (-1)^n + \frac{(-1)^{n+1}}{n} \right\}$$

(a) Are A, B and C bounded or unbounded in $\mathbb{R}$?

(b) Find, if they exist, the infimum, supremum, maximum and minimum of A, B and C.

1.3 Consider the following subsets of real numbers

$$A = [-1, 10] \cup \{14\} \qquad B = \{-1\} \cup [0, 10] \cup \{12\} \qquad C = (-\infty, 10)$$

(a) Are A, B and C compact and convex sets? Find the boundary sets of A, B and C. Is $A \cup B$ closed, bounded and convex?

(b) Is C complement (with respect to $\mathbb{R}$) closed, not bounded and convex?

(c) Describe the topological properties of -1 with respect to A, B and C.

1.4 Let $A = \left\{x \in \mathbb{R}^2 : \ x_1^2 - x_2 \leq 4\right\}$ be a subset of $\mathbb{R}^2$.

(a) Represent A on $\mathbb{R}^2$ and indicate some topological properties of A.

(b) Find the following sets: $\operatorname{int}(A)$ (the set of interior points of A), ∂A (the set of boundary points of A) e A' (the set of accumulation points of A).

(c) Find the set of accumulation points of A^c.

1.5 Consider the following set $A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9, \ |y| \leq 3\right\}$.

(a) Show that $\mathbf{x}^* = (-2, 2) \in A$ and $\mathbf{z}^* = (-2, 3) \notin A$. Is the set A open? Closed? Bounded? Compact?

(b) Write the neighborhood of the point $\mathbf{x}^* = (-2, 2)$ with radius 1 (that is $B_1(\mathbf{x}^*)$). Is $\mathbf{w}^* = (0, -3)$ a boundary point of A? Is it an accumulation point of A?

(c) Write the neighborhood of the point $\mathbf{x} = \mathbf{0}$ with radius 5 (that is $B_5(\mathbf{0})$).

1.6 Let A, B, C and D be subsets of $\mathbb{R}^2$ defined as follows

$$A = \{x \in \mathbb{R}^2 : \ x_1 x_2 \geq 0\} \qquad B = \{x \in \mathbb{R}^2 : \ \|\mathbf{x}\| \leq 2\}$$

$$C = \{x \in \mathbb{R}^2 : \ |x_1| \leq 1\} \qquad D = \{x \in \mathbb{R}^2 : \ \|\mathbf{x}\| > 1\}$$

Establish some topological properties of the sets $A \cap B$, $A \cap B \cap C$, $B \cap D$ and $A^c \cap C^c$.

1.1 Proceed by contradiction. Assume that $x > 0$ and take $\varepsilon = x/2$.

1.2 **(a)** We have

$$A = (-\infty, a) \cup (b, c) \quad x_B = \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases} \quad x_c = \begin{cases} 2 - 1/n & n \text{ even} \\ 1/n & n \text{ odd} \end{cases}$$

A is unbounded and B, C are bounded; **(b)** $\sup(A) = c$, $\inf(A) = -\infty$; $\sup(B) = 2$, $\inf(B) = 0$; $\sup(C) = 2$, $\inf(C) = 0$.

1.3 **(a)** A is closed and bounded (therefore compact), but it is not convex; B is closed and bounded (and compact), but it has no the convexity property; C is open and not bounded (and not compact), but is convex; the set of boundary points of A is $\{-1, 10, 14\}$, the set of boundary points of B is $\{-1, 0, 10, 12\}$, the set of boundary points of C is the singleton $\{10\}$. The union of A and B is defined by

$$A \cup B = [-1, 10] \cup \{12\} \cup \{14\} \Rightarrow A \cup B \text{ is closed, bounded and not convex}$$

(b) $C^c = \mathbb{R} \setminus C = [10, +\infty)$ is closed, not bounded and convex; **(c)** -1 is a boundary point (and an accumulation point) of A, an isolated point (and a boundary point) of B and an internal point (and an accumulation point) of C.

1.4 **(a)** A is closed, not bounded and convex. A is not compact; **(b)** The sets required are

$$\text{int}(A) = \{x \in \mathbb{R}^2 : \ x_1^2 - 4 < x_2\}; \qquad \partial A = \{x \in \mathbb{R}^2 : \ x_1^2 - 4 = x_2\}$$

and $A' = \{x \in \mathbb{R}^2 : \ x_1^2 - 4 \leq x_2\}$; **(c)** $(A^c)' = \{x \in \mathbb{R}^2 : \ x_1^2 - 4 \geq x_2\}$.

1.5 **(a)** By substitution we have $\mathbf{x}^* = (-2, 2) \in A$ and $\mathbf{z}^* = (-2, 3) \notin A$; A is closed and bounded ($\Rightarrow A$ compact); **(b)** We have $B_1(\mathbf{x}^*) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{x}^*\| < 1\}$. Yes, $\mathbf{w}^* = (0, -3) \in A$ is a boundary point and an accumulation point of A; **(c)** We have

$$B_5(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{0}\| < 5\} = \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 < 25\}$$

1.6 $A \cap B$ is closed, bounded, compacy and not convex; $A \cap B \cap C$ is closed, bounded, compact and not convex; $B \cap D$ is bounded; $A^c \cap C^c$ is open, not bounded and not convex.