Mathematics (BEMACS) - A. Y. 2019/20

Exercises - Set 5

1.1 Find the range $f(A)$ for each of the following functions $f : A \subseteq \mathbb{R} \to \mathbb{R}$:

(a) $f(x) = \sqrt[3]{1-x}$ (b) $f(x) = 2|x| + 2$ (c) $f(x) = \sqrt{1+\sqrt{x}}$

(d) $f(x) = 2 - (x-1)^2$ (e) $f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases}$ (f) $f(x) = 2\sqrt{x} - 1$

1.2 Consider the functions $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ (where A_f and A_g are their natural domains) defined by:

(a) $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x+1}$ $g(x) = \dfrac{2x}{x^2-1}$ (b) $f(x) = \sqrt{\dfrac{x-2}{x-3}}$ $g(x) = \dfrac{\sqrt{x-2}}{\sqrt{x-3}}$

Are f and g equal? In the case of $f \neq g$, find the biggest subset of $\mathbb{R}$ such that $f = g$.

1.3 Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x - 1$ and $g(x) = x^2 - 2x$. Solve $(g \circ f \circ f)(x) = 0$.

1.4 Consider the following functions $f, g, h, t : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = x + a \qquad g(x) = \sqrt[3]{x} + 4 \qquad h(x) = e^{2x} + 2 \qquad t(x) = a - \sqrt{2x}$$

with a real number.

(a) Does the following condition $h(\mathbb{R}) \subset f(\mathbb{R}) = g(\mathbb{R})$ hold among the ranges of the functions f, g, h?

(b) Using the definition, show that the functions f, g, h are strictly increasing (in their natural domains).

(c) Using the definition, show that the function t is strictly decreasing (in its natural domain).

1.5 Let $f : A_f \subseteq \mathbb{R} \to \mathbb{R}$ and $g : A_g \subseteq \mathbb{R} \to \mathbb{R}$ be functions with $f(A_f) \subseteq A_g$. Show that if f is even then $g \circ f$ is also even.

1.6 Consider the following function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ defined by

(a) $f(x) = x^3$ (b) $f(x) = x^2$ (c) $f(x) = \sqrt[3]{2x} - 1$

(d) $f(x) = \ln(x-2)$ (e) $f(x) = e^{3x} - 1$ (f) $f(x) = 1 - |4x|$

Which function, using the definition, is bijective on its natural domain?

1.1 (a) $\mathbb{R}$; (b) $[2, +\infty)$; (c) $[1, +\infty)$; (d) $(-\infty, 2]$; (e) $[0, +\infty)$; (f) $[1, +\infty)$.

1.2 (a) f, g are equal; (b) $A_g \subset A_f$, f and g are different. In $(3, +\infty)$, f and g are equal.

1.3 (a) $\sqrt[3]{3/4}$; (b) $-1/2$; (c) 1; (d) We have

$$f\left[f\left(\frac{2}{3}\right)\right] = f\left[\frac{1}{(2/3)-1}\right] = f\left[\frac{1}{-1/3}\right] = f\left[-3\right] = \frac{1}{(-3)-1} = -\frac{1}{4}$$

(e) It does not exist; (f) We have

$$\begin{aligned} g\left[f\left[g\left(\sqrt{9}\right)\right]\right] &= g\left[f\left[\sqrt[3]{1-9}\right]\right] = g\left[f\left[-\sqrt[3]{8}\right]\right] = g\left[f\left[-2\right]\right] = \\ &= g\left[\frac{1}{-2-1}\right] = g\left[-\frac{1}{3}\right] = \sqrt[3]{1-\left(-\frac{1}{3}\right)^2} = \sqrt[3]{-\frac{8}{9}} = -\frac{2}{\sqrt[3]{9}} \end{aligned}$$

1.4 The functions f, g and h are defined for any real x. The ranges are $f\left(\mathbb{R}\right) = \mathbb{R}$, $g\left(\mathbb{R}\right) = \mathbb{R}$ and $h\left(\mathbb{R}\right) = (2, +\infty)$ consequently $h\left(\mathbb{R}\right) \subset f\left(\mathbb{R}\right) = g\left(\mathbb{R}\right)$. (a) For any $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$ we have

$$x_1 < x_2 \quad \Rightarrow \quad x_1 < x_2 \quad \Rightarrow \quad x_1 + a < x_2 + a \quad \Rightarrow \quad f\left(x_1\right) < f\left(x_2\right)$$

therefore f is strictly increasing (so strictly monotonic) in $\mathbb{R}$. Carry on in the same way for functions g and h. (c) $A = [0, +\infty)$. Carry on as point (b).

1.5 $(f \circ g)$ is defined on $A_{g\circ f} = A_f$, therefore $x \in A_{g\circ f}$ and $-x \in A_{g\circ f}$. Let's start from the assumption that f is an even function on its natural domain A_f, we have

$$f\left(-x\right) = f(x) \quad \Rightarrow \quad g\left[f(-x)\right] = g\left[f(x)\right] \quad \Rightarrow \quad \left(g \circ f\right)\left(-x\right) = \left(g \circ f\right)\left(x\right)$$

therefore $\left(g \circ f\right)$ is also an even function on $A_{g\circ f}$.

1.6 (a) Tes; (b) No; (c) Yes; (d) Yes; (e) No; (e) No.