Mathematics (BEMACS) - A. Y. 2019/20

Exercises - Set 6

1.1 Consider the following function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$(1)\ f(x) = \begin{cases} -x & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases} \qquad (2)\ f(x) = \begin{cases} x^3 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

(a) Draw the graph of the function f (using the graphs of the elementary functions). Answer the following questions using the graph G_f.

(b) Is f injective, surjective and bijective? Is f bounded?

(c) Is f monotonic? How many solutions has the equation $f(x) = 16$?

1.2 Let $x \in \mathbb{R}$ be a real number. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

(a) $f(x) = \max[\min(x, 0), x] - \min[\max(x, 0), 0]$;

(b) $f(x) = 1 - \max[\max(|x|; 0); x]$

Draw the graph of f and indicate some properties of f.

1.3 Let $A = \{x \in \mathbb{R}^2 :\ x_1^2 - x_2 \leq 4\}$ be a subset of $\mathbb{R}^2$.

(a) Represent A on $\mathbb{R}^2$ and indicate some topological properties of A.

(b) Find the following sets: $\text{int}(A)$ (the set of interior points of A), ∂A (the set of boundary points of A) and A' (the set of accumulation points of A).

(c) Find the set of accumulation points of A^c.

1.4 Find the inverse f^{-1} for these functions:

$$(a)\ f(x) = \begin{cases} x & x \geq 0 \\ -x^2 & x < 0 \end{cases} \qquad (b)\ f(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ x^3 & x < 0 \end{cases}$$

1.5 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ |x| - 1 & |x| > 1 \end{cases}$$

Find, if they exist, the local and global maximizers and minimizers. (Help yourself by a graphical representation of f).

1.6 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x^4$.

(a) Does f have a global minimizer? If so, what is it?

(b) Consider $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^3 + 1$, find, if it exists, the composite function $(g \circ f)$.

(c) Does $(g \circ f)$ have a global minimizer?

1.1 (1) **(b)** f is neither injective or surjective (f is not bijective). f is not bounded; **(c)** f is not monotonic. We have 2 solutions. (2) **(b)** f is injective but not surjective (f is not bijective). f is not bounded; **(c)** f is monotonic. We have 1 solution.

1.2 Let $x \in \mathbb{R}$ be a real number. **(a)** We can consider:

- if $x \geq 0$, $\min(x, 0) = 0$ and $\max(x, 0) = x$, by substitution we have

$$f(x) = \max[0, x] - \min[x, 0] = x - 0 = x$$

- if $x < 0$, $\min(x, 0) = x$ and $\max(x, 0) = 0$, by substitution we have

$$f(x) = \max[x, x] - \min[0, 0] = x - 0 = x$$

Therefore f is equal to

$$f(x) = \max[\min(x, 0), x] - \min[\max(x, 0), 0] = x$$

On $\mathbb{R}$ we have that f is monotone (strictly monotone), convex and unbounded. **(b)** We have

$$f(x) = 1 - \max[\max(|x|; 0); x] = 1 - |x| = \begin{cases} 1 - x & x \geq 0 \\ 1 + x & x < 0 \end{cases}$$

On $\mathbb{R}$ we have that f is not monotone. f is convex and unbounded.

1.3 **(a)** A is closed, not bounded and convex. A is not compact; **(b)** The sets required are

$$\operatorname{int}(A) = \left\{x \in \mathbb{R}^2 : \ x_1^2 - 4 < x_2\right\}; \qquad \partial A = \left\{x \in \mathbb{R}^2 : \ x_1^2 - 4 = x_2\right\}$$

and $A' = \left\{x \in \mathbb{R}^2 : \ x_1^2 - 4 \leq x_2\right\}$; **(c)** $(A^c)' = \left\{x \in \mathbb{R}^2 : \ x_1^2 - 4 \geq x_2\right\}$.

1.4 We have:

$$\text{(a) } f^{-1}(y) = \begin{cases} y & y \geq 0 \\ -\sqrt{-y} & y < 0 \end{cases} ; \qquad \text{(b) } f^{-1}(y) = \begin{cases} y^2 & y \geq 0 \\ \sqrt[3]{y} & y < 0 \end{cases}$$

1.5 0 is a local maximizer with $f(0) = 1$, and -1, 1 are global minimizers with $f(-1) = f(1) = 0$.

1.6 **(a)** 0 is a strong global minimizer of f with value $f(0) = 0$; **(b)** $(g \circ f)$ exists and

$$(g \circ f)(x) = g[f(x)] = g\left(4x^4\right) = 64x^{12} + 1$$

(c) 0 is a strong global minimizer of $(g \circ f)$, because g is strictly increasing on $\mathbb{R}$.