Mathematics (BEMACS) - A. Y. 2019/20

Exercises - Set 7

1.1 (a) Consider the following limits of functions:

$$(1)\ \lim_{x\to 0}\left(\frac{1}{x}\right)^3 \qquad (2)\ \lim_{x\to 0}\frac{|x|}{x} \qquad (3)\ \lim_{x\to 0} e^{1/x} \qquad (4)\ \lim_{x\to 0} e^{-2/x^3}$$

Show that the limits do not exist. (b) Calculate

$$\lim_{x\to 0}\left(\sin x\cdot\sin\frac{1}{x}\right)$$

1.2 (a) Let a be a positive real number. Consider $f:\mathbb{R}\to\mathbb{R}$ the function defined by

$$f(x)=\frac{a^{x+2}+2^x}{a^x+2^{x+1}}$$

Calculate the limit of $f(x)$ as $x\to+\infty$. (b) Calculate the following limits:

$$(1)\ \lim_{x\to\frac{\pi}{2}^-}\frac{1}{\ln(\tan x+1)};\ (2)\ \lim_{x\to+\infty}\sqrt[3]{\ln\left(1-2^{1-x}\right)};\ (3)\ \lim_{x\to 0^+}\frac{1+\ln x}{x^3};\ (4)\ \lim_{x\to+\infty}\sqrt[3]{\frac{1}{2-\ln x}}$$

1.3 For which values of $k\in\mathbb{R}$ are the functions $f:A\subseteq\mathbb{R}\to\mathbb{R}$ defined by:

$$\text{(a)}\quad f(x)=\begin{cases}\left(kx^2+7k\right)^{-1/2} & \text{if } x<1\\ k & \text{if } x\geq 1\end{cases} \qquad \text{(b)}\quad f(x)=\begin{cases}\dfrac{1-\sqrt{3x-2}}{x-1} & \text{if } x>1\\ 2k+1 & \text{if } x=1\end{cases}$$

$$\text{(c)}\quad f(x)=\begin{cases}\dfrac{\sin\left(2x^2\right)}{x^2} & \text{if } x\neq 0\\ k & \text{if } x=0\end{cases} \qquad \text{(d)}\quad f(x)=\begin{cases}\dfrac{x^2-x}{\sqrt{3+x^2}-2} & \text{if } x\neq 1\\ k & \text{if } x=1\end{cases}$$

continuous in their domains?

1.4 Consider the functions $f,g:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2+x$ and $g(x)=x^3$. Can we say that $g=o(f)$ when $x\to 0$?

1.5 Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}ax-1 & x<0\\ x^2-bx-a & 0\leq x<1\\ 2e^{x-1}-ax+b+1 & x\geq 1\end{cases}$$

Find a and b such that f is continuous on $\mathbb{R}$.

1.6 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{se} \quad x \neq 0 \\ \\ a & \text{se} \quad x = 0 \end{cases}$$

Is f continuous for any x real number?

1.1 (a) (1) We have

$$\lim_{x \to 0^-} \left(\frac{1}{x}\right)^3 = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \left(\frac{1}{x}\right)^3 = +\infty \quad \Rightarrow \quad \lim_{x \to 0} \left(\frac{1}{x}\right)^3 \text{ does not exist}$$

Do the same for cases (2), (3) and (4). **(b)** We know that

$$0 \leq \left|\sin x \cdot \sin \frac{1}{x}\right| \leq |\sin x| \cdot \left|\sin \frac{1}{x}\right| \leq |\sin x|$$

therefore the limit required is equal to 1.

1.2 (a) We have three cases: if $a > 2$ then $f(x) \to a^2$ as $x \to +\infty$, if $0 < a < 2$ then $f(x) \to 1/2$ as $x \to +\infty$ and if $a = 2$ then $f(x) \to 5/3$ as $x \to +\infty$. **(b)** (1) 0^+; (2) 0^-; (3) $-\infty$; (4) 0^-.

1.3 (a) $1/2$; **(b)** $-5/4$; **(c)** 2; **(d)** 2.

1.4 We have

$$\lim_{x \to 0} \frac{g(x)}{f(x)} = \lim_{x \to 0} \frac{x^3}{x^2 + x} = \lim_{x \to 0} \frac{x^2}{x+1} = 0 \Rightarrow g = o(f) \text{ as } x \to 0$$

1.5 $a = 1$, $b = -1$.

1.6 We know that $\lim_{x \to 0} f(x) = 1$ and $f(0) = a$. The function is continuous on $\mathbb{R}$ if and only if $a = 1$.