Mathematics (BEMACS) - A. Y. 2019/20

Exercises - Set 8

1.1 Find the difference quotient at the point x_0 (with respect to the increment h) for each of the following functions $f : A \subseteq \mathbb{R} \to \mathbb{R}$ defined by:

$$\text{(a)} \begin{cases} f(x) = x^2 - x \\ x_0 \in \mathbb{R} \end{cases} \qquad \text{(b)} \begin{cases} f(x) = e^{x-1} \\ x_0 \in \mathbb{R} \end{cases} \qquad \text{(c)} \begin{cases} f(x) = \ln(1+x) \\ x_0 \in (0, +\infty) \end{cases}$$

1.2 (a) Calculate, using the definition, the derivative at the points x_0 of the function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ defined by:

$$(1)\ f(x) = 2x^2 + x \quad (2)\ f(x) = \frac{1}{x+1} \quad (3)\ f(x) = \ln(1+x)$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a function defined by $g(x) = \sqrt{|-2x|}$. Is g derivable at any point of $\mathbb{R}$?

1.3 Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined by:

$$\text{(a)}\ f(x) = \frac{x}{x^2+1} \qquad \text{(b)}\ f(x) = \frac{1}{x-1} - \frac{4}{x^2} \qquad \text{(c)}\ f(x) = \ln\left(\frac{x-1}{x+1}\right)$$

$$\text{(d)}\ f(x) = (x-2)^2 e^{3x} \qquad \text{(e)}\ f(x) = (x \ln x)^2 \qquad \text{(f)}\ f(x) = [\ln(2x-1)]^4$$

Calculate the derivative of f.

1.4 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} (x+1)^2 & \text{if} \quad -2 \le x < 0 \\ 1 - x^2 & \text{if} \quad 0 \le x \le 1 \end{cases}$$

(a) Calculate, using the definition, the following derivatives:

$$(1)\ f'_+(-2) \qquad (2)\ f'(-1) \qquad (3)\ f'_-(0) \qquad (4)\ f'_+(0) \qquad (5)\ f'(1/2) \qquad (6)\ f'_-(1)$$

(b) Is f at the point $x_0 = 0$ continuous and differentiable?

1.5 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = kx^2$.

(a) Calculate, using the definition, the derivative of f at x_0. Is f continuous at x_0?

(b) Find the equation of the tangent line to the graph of f at x_0.

1.6 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = e^{\alpha x + \beta}$, with $\alpha, \beta > 0$.

(a) Calculate, using the definition, the derivative of f at x_0.

(b) Find the equation of the tangent line to the graph of f at x_0.

1.1 (a) $A=\mathbb{R}$. The difference quotient of f in the interval $[x_0, x_0+h]$, with $h \neq 0$, is

$$\frac{\Delta f}{\Delta x}=\frac{\left[(x_0+h)^2-(x_0+h)\right]-\left(x_0^2-x_0\right)}{h}=\frac{(2x_0-1)\,h+h^2}{h}=(2x_0-1)+h$$

(b) $A=\mathbb{R}$ (the function is defined for any real x). We have

$$\frac{\Delta f}{\Delta x}=\frac{f(x_0+h)-f(x_0)}{h}=\frac{e^{(x_0+h)-1}-e^{x_0-1}}{h}=e^{x_0-1}\frac{e^h-1}{h}$$

(c) $A=(-1,+\infty)$. We have

$$\frac{\Delta f}{\Delta x}=\frac{f(x_0+h)-f(x_0)}{h}=\frac{\ln[1+(x_0+h)]-\ln(1+x_0)}{h}=\frac{1}{h}\ln\left(1+\frac{h}{1+x_0}\right)$$

1.2 (a) (1) The limit of the difference quotient of f in the interval $[x_0, x_0+h]$, with $h \neq 0$, for $h \to 0$ is

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{\left[2(x_0+h)^2+(x_0+h)\right]-\left[2x_0^2+x_0\right]}{h} \\ &= \lim_{h\to 0}\frac{(4x_0+1)\,h+2h^2}{h}=\lim_{h\to 0}(4x_0+1+2h)=4x_0+1 \end{aligned}$$

therefore f is derivable at x_0 and $f'(x_0)=4x_0+1$; (2) We have

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{(x_0+h)+1}-\dfrac{1}{x_0+1}}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x_0+h)+1}-\frac{1}{x_0+1}\right]=\lim_{h\to 0}\frac{1}{h}\frac{-h}{[(x_0+h)+1](x_0+1)} \\ &= -\lim_{h\to 0}\frac{1}{[(x_0+h)+1](x_0+1)}=-\frac{1}{(x_0+1)^2} \end{aligned}$$

(3) $f'(x_0)=1/(1+x_0)$. **(b)** Let $x_0 \in \mathbb{R}$. We have

$$\begin{aligned} \lim_{h\to 0}\frac{\Delta g}{\Delta x} &= \lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}=\lim_{h\to 0}\frac{\sqrt{|-2(x_0+h)|}-\sqrt{|-2x_0|}}{h} \\ &= \sqrt{2}\lim_{h\to 0}\frac{\sqrt{|(x_0+h)|}-\sqrt{|x_0|}}{h}=\sqrt{2}\lim_{h\to 0}\frac{|(x_0+h)|-|x_0|}{h\left(\sqrt{|(x_0+h)|}+\sqrt{|x_0|}\right)} \end{aligned}$$

and for any $x_0 \neq 0$ function g is derivable with $g'(x)=1/\sqrt{2\,|x_0|}$. At $x_0=0$, function g is not derivable (we have a cuspidal point).

1.3 (a) We have

$$f'(x) = \left(\frac{x}{x^2+1}\right)' = \frac{x' \cdot \left(x^2+1\right) - x \cdot \left(x^2+1\right)'}{\left(x^2+1\right)^2} = \frac{1-x^2}{\left(x^2+1\right)^2}$$

(b) $\dfrac{8}{x^3} - \dfrac{1}{(x-1)^2}$; **(c)** $\dfrac{2}{x^2-1}$; **(d)** $(x-2)\, e^{3x}\, (3x-4)$; **(e)** $2\,(x \ln x)\,(\ln x + 1)$; **(f)** $\dfrac{8\,[\ln(2x-1)]^3}{2x-1}$.

1.4 (a) (1) The limit of the difference quotient of f in the interval $[-2, -2+h]$, with $h > 0$, for $h \to 0^+$ is

$$\lim_{h\to 0^+} \frac{\Delta f}{\Delta x} = \lim_{h\to 0^+} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0^+} \frac{(h-1)^2 - 1}{h} = \lim_{h\to 0^+} (h-2) = -2$$

therefore $f'_+(-2) = -2$ (f is differentiable from the right at -2). (2) $f'(-1) = 0$; (3) $f'_-(0) = 2$; (4) $f'_+(0) = 0$; (5) $f'(1/2) = -1$; (6) The limit of the difference quotient of f in the interval $[1+h, 1]$, with $h < 0$, for $h \to 0^-$ is

$$\lim_{h\to 0^-} \frac{\Delta f}{\Delta x} = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{1-(h+1)^2}{h} = -\lim_{h\to 0^-} (h+2) = -2$$

therefore we get $f'_-(1) = -2$ (f is differentiable from the left at 1); **(b)** The function f is continuous in $(0, 1)$, in fact we get

$$\lim_{x\to 0^-} (x+1)^2 = 1 = \lim_{x\to 0^+} \left(1-x^2\right) \quad \Rightarrow \quad \lim_{x\to 0} f(x) = 1 = f(0)$$

but it's not differentiable, given that we have $f'_-(0) = 2 \neq f'_+(0) = 0$ (0 is a corner point).

1.5 (a) $f'(x) = 2kx_0$; **(b)** $y = 2kx_0 x - kx_0^2$.

1.6 (a) $f'(x) = \alpha e^{\alpha x_0 + \beta}$; **(b)** We have

$$y = \alpha e^{\alpha x_0 + \beta} x + e^{\alpha x_0 + \beta}(1 - \alpha x_0)$$