TNM Exercise 5: Variational Bayes

Saurabh Vaishampayan (svaishampaya@student.ethz.ch)
Linus Rüttimann (rlinus@student.ethz.ch)
Elia Torre (etorre@student.ethz.ch)
Timothy Kurer (tkurer@student.ethz.ch)

Group **H**
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Ex 5.1

We include the scans for handwritten solutions to all problems of Ex5.1 below

TNU Ex 5

1. $y = \mu + \varepsilon$, $\varepsilon \sim N(\varepsilon; \frac{1}{\tau})$

$y \sim N(\mu; \frac{1}{\tau})$

$$p(y|\mu,\tau) = \frac{1}{\sqrt{2\pi \cdot \frac{1}{\tau}}} e^{-\frac{(\mu-y)^2}{2}\tau}$$

$$\log p(y|\mu,\tau) = -\frac{\tau}{2}\frac{(y-\mu)^2}{2} + \log\sqrt{\frac{\tau}{2\pi}}$$

$$= \frac{1}{2}\log\tau - \frac{\tau}{2}\frac{(y-\mu)^2}{2} - \frac{1}{2}\log 2\pi.$$

$\underline{y} = (y_1, y_2 \dots y_N)^T$

$$\log p(\underline{y}|\mu,\tau) = \log \prod_{i=1}^{N} p(y_i|\mu,\tau) = \sum_{i=1}^{N} \log p(y_i|\mu,\tau)$$

$$= \sum_{i=1}^{N}\left(\frac{1}{2}\log\tau - \frac{\tau}{2}(y_i-\mu)^2 - \frac{1}{2}\log 2\pi\right)$$

$$= \frac{N}{2}\log\tau - \frac{N}{2}\log 2\pi - \frac{\tau}{2}\sum_{i=1}^{N}(y_i-\mu)^2$$

b) $p(\theta) = \sqrt{\frac{\lambda_0 \tau}{2\pi}} \, e^{-\frac{\lambda_0 \tau}{2}(\mu-\mu_0)^2} \, \frac{b_0^{a_0} \tau^{a_0-1}}{\Gamma(a_0)} e^{-b_0 \tau}.$

$$\log p(\theta) = -\frac{\lambda_0 \tau}{2}(\mu-\mu_0)^2 - b_0\tau + \frac{1}{2}\log\frac{\tau}{2\pi} + \frac{1}{2}\log\lambda_0$$
$$+ a_0 \log b_0 + (a_0-1)\log\tau - \log\Gamma(a_0)$$

$$\log p(y,\theta) = \log p(\theta)p(y|\theta) = \log p(\theta) + \log p(y|\theta)$$

$$= -\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2 - \frac{\tau}{2}\sum_{i=1}^{n}(\mu-y_i)^2 + \frac{(N+1)}{2}\log\frac{\tau}{2\pi}$$
$$+ (a_0-1)\log\tau - b_0\tau + \underbrace{\frac{1}{2}\log\lambda_0 + a_0\log b_0 - \log\Gamma(a_0)}_{\text{const}}$$

c) $q(\theta) = q(\mu)\, q(\tau)$

We know that under mean field approximation, optimal approximate posterior for k^{th} subset of param.

$$\ln q_k^*(\theta_k) = E[\ln p(\theta,y)]_{q_1,\dots q_{k-1},q_{k+1},\dots q_M} + \text{const}$$

To find $q^*(\mu)$

$$\ln q^*(\mu) = E[\ln p(\theta,y)]_\tau + \text{const}$$

$$E_\tau[\ln p(\theta,y)] = E_\tau\Big[-\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2 - \frac{\tau}{2}\sum_{i=1}^{n}(\mu-y_i)^2 + \frac{(N+1)}{2}\log\frac{\tau}{2\pi}$$
$$+(a_0-1)\log\tau - b_0\tau + \frac{1}{2}\log\lambda_0 + a_0\log b_0 - \log\Gamma(a_0)\Big]$$

$$= E_\tau \Big[\underbrace{-\frac{\lambda_0 \tau}{2}(\mu-\mu_0)^2 - \frac{\tau}{2}\sum_{i=1}^{n}(\mu-y_i)^2}_{\mathrm{I}} + \frac{(N+1)}{2}\log\frac{\tau}{2\pi} + (a_0-1)\log\tau \underbrace{- b_0\tau + \text{const}}_{\mathrm{II}} \Big].$$

$E_\tau[\mathrm{I}]$ will yield ~~a~~ function of μ.

$E_\tau[\mathrm{II}]$ is a constant function wrt μ.

$$\therefore \; E_\tau[\ln p(\theta, y)] = E_\tau\Big[-\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2 - \frac{\tau}{2}\sum_{i=1}^{n}(\mu-y_i)^2\Big] + \text{const}$$

$$= -\frac{1}{2}E_\tau[\tau]\Big(\sum_{i=1}^{n}(y_i-\mu)^2 \cancel{\;} + \lambda_0(\mu-\mu_0)^2\Big) + \text{const}$$

$$\ln q^*(\mu) = E_\tau[\ln p(\theta, y)] + \text{const}$$

$$= -\frac{1}{2}E_\tau[\tau]\Big(\sum_{i=1}^{n}(y_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\Big) + \text{const}$$

d)

$$\ln q^*(\mu) = -\frac{1}{2}\underbrace{E_\tau[\tau]}_{\bar{\tau}}\Big(\sum_{i=1}^{n}(y_i-\mu)^2 + \lambda_0(\mu-\mu_0)^2\Big) + \text{const}$$

$$\sum_{i=1}^{n}(y_i-\mu)^2 = N\mu^2 - 2\mu\sum_{i=1}^{N}y_i + \sum_{i=1}^{N}y_i^2$$

$$\cancel{= N\mu^2}$$

$$\lambda_0(\mu-\mu_0)^2 = \lambda_0\mu^2 - 2\lambda_0\mu_0\mu + \lambda_0\mu_0^2$$

$$\therefore \ln q^*(\mu) = -\frac{\bar{\tau}}{2}\left[N\mu^2 + \lambda_0\mu^2 - 2\lambda_0\mu_0\mu - 2\mu\sum_{i=1}^{N} y_i + \sum_{i=1}^{N} y_i^2 + \lambda_0\mu_0^2\right] + \text{const}$$

$$= -\frac{\bar{\tau}}{2}\left[(N+\lambda_0)\mu^2 - 2\mu(\lambda_0\mu_0 + N\bar{y})\right] + \text{const}$$

$$= -\frac{\bar{\tau}}{2}(N+\lambda_0)\mu^2 + \bar{\tau}\mu(\lambda_0\mu_0 + N\bar{y})$$

e) To check if $\ln q^*(\mu)$ corresponds to $q^*(\mu)$ Gaussian

We assume $q^*(\mu) \sim N(m, s^2)$.

If $\ln \mathcal{N}(m, s^2)$ indeed matches the above ~~then~~ for some m, s^2, then $q^*(\mu)$ is gaussian with $\mathcal{N}(m, s^2)$

$$\ln \mathcal{N}(\mu; m, s^2) = \ln \frac{1}{\sqrt{2\pi}\, s} e^{-\frac{(\mu-m)^2}{2s^2}}$$

$$= -\frac{(\mu-m)^2}{2s^2} - \ln s - \frac{1}{2}\ln 2\pi.$$

~~In eq (7)~~ $$= -\frac{1}{2s^2}\mu^2 + \frac{m}{s^2}\mu.$$

Matching coefficients with eq 7 from exercise sheet.

$$\frac{1}{s^2} = \bar{\tau}(N+\lambda_0) \quad \Rightarrow \quad s^2 = \frac{1}{\bar{\tau}(N+\lambda_0)}$$

$$\frac{m}{s^2} = \bar{\tau}(\lambda_0\mu_0 + N\bar{y}) \qquad m = \frac{\lambda_0\mu_0 + N\bar{y}}{N+\lambda_0}$$

f) As before, under mean-field approx we have

$$\ln q^*(\tau) = E_\mu[\ln p(\theta, y)] + \text{const}$$

$$= E_\mu\left[-\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2 - \frac{\tau}{2}\sum_{i=1}^{N}(\mu-y_i)^2 + \frac{(N+1)}{2}\log\frac{\tau}{2\pi}\right.$$
$$\left. + (a_0-1)\log\tau - b_0\tau + \text{const}\right] + \text{const}$$

$$= -\frac{\tau}{2}E_\mu\left[\sum_{n=1}^{N}(y_n-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right]$$
$$+ E_\mu\left[\underbrace{\frac{(N+1)}{2}\log\tau + (a_0-1)\log\tau - b_0\tau}_{\text{independent of }\mu}\right] + \text{const}$$

$$\boxed{\ln q^*(\tau) = -\frac{\tau}{2}E_\mu\left[\sum_{n=1}^{N}(y_n-\mu)^2 + \lambda_0(\mu-\mu_0)^2\right] + \left[\frac{(N+1)}{2} + (a_0-1)\right]\log\tau - b_0\tau + \text{const}}$$

g) $$E_\mu\left[\sum_{n=1}^{N}(y_n-\mu)^2\right] = \sum_{n=1}^{N}E[(y_n-\mu)^2]$$
$$= \sum_{n=1}^{N}\left[(y_n-m)^2 + s^2\right] = \sum_{n=1}^{N}(y_n-m)^2 + Ns^2$$

$$E_\mu[\lambda_0(\mu-\mu_0)^2] = \lambda_0\left[(\mu_0-m)^2 + s^2\right]$$

$$\therefore \ln q^*(\tau) = -\frac{\tau}{2}\left[\sum_{n=1}^{N}(y_n-m)^2 + \lambda_0(\mu_0-m)^2 + (N+\lambda_0)s^2\right]$$
$$- b_0\tau + \left(a_0 + \frac{N+1}{2} - 1\right)\log\tau + \text{const}.$$

h) $\mathrm{Gam}(\tau|a,b) = \dfrac{b^a \tau^{a-1}}{\Gamma(a)} e^{-b\tau}$

As before we compare $\log \mathrm{Gam}(\tau|a,b)$ to $\log q^*(\tau)$ and see if they match for some a, b.

$$\log \mathrm{Gam}(\tau|a,b) = a\log b + (a-1)\log\tau - b\tau - \log\Gamma(a).$$

$$\log q^*(\tau) = -\tau\left[b_0 + \frac{1}{2}\left(\sum_{n=1}^{N}(y_n-m)^2 + \lambda_0(\mu_0-m)^2 + (N+\lambda_0)s^2\right)\right]$$

$$+ \left(a_0 + \frac{N+1}{2} - 1\right)\log\tau + \text{const.}$$

Matching coeff for τ and $\log\tau$

$$-b = -\left(b_0 + \frac{1}{2}\left(\sum_{n=1}^{N}(y_n-m)^2 + \lambda_0(\mu_0-m)^2 + (N+\lambda_0)s^2\right)\right)$$

$$a-1 = a_0 + \frac{N+1}{2} - 1 \Rightarrow a = a_0 + \frac{N+1}{2}.$$

$$b = b_0 + \frac{1}{2}\left(\sum_{n=1}^{N}(y_n-m)^2 + \lambda_0(\mu_0-m)^2 + (N+\lambda_0)s^2\right)$$

i) $\mathcal{F} = E_q[\ln p(y|\theta)] - KL(q(\theta)||p(\theta))$

$E_q[\ln p(y|\theta)] = \cancel{E_{\mu\tau} \int q(\mu)q(\tau) \ln p(y|\theta)}\, d.$

$= \int q(\mu)q(\tau) \log p(y|\mu,\tau)\, d\mu\, d\tau.$

$= \int q(\mu)q(\tau) \left[\frac{N}{2}\log\tau - \frac{\tau}{2}\sum_{n=1}^{N}(y_n-\mu)^2 - \frac{N}{2}\log 2\pi \right] d\mu d\tau.$

$= \frac{N}{2}\int q(\tau)\log\tau\, d\tau - \frac{N}{2}\log 2\pi - \int\left(\int q(\tau)\tau\, d\tau\right)\frac{1}{2}\left(\sum_{n=1}^{N}(y_n-\mu)^2\right) d\mu$

$= \frac{N}{2} E_\tau[\log\tau] - \frac{N}{2}\log 2\pi - E_\tau[\tau]\, E_\mu\left[\frac{1}{2}\sum_{n=1}^{N}(y_n-\mu)^2\right]$

$KL(q(\theta)||p(\theta)) = \int q(\mu)q(\tau) \ln \frac{q(\mu) \ln q(\tau)}{p(\mu,\tau)}\, d\mu\, d\tau.$

$= \int q(\mu)q(\tau) \ln q(\tau) q(\mu)\, d\mu d\tau$

$+ \int q(\mu)q(\tau) \ln p(\tau,\mu)\, d\mu, d\tau$

$= E_\mu[\log q(\mu)] + E_\tau[\log q(\tau)]$

$+ \int q(\mu)q(\tau) \ln p(\tau,\mu)\, d\mu\, d\tau$

$$\int q(\theta) \ln p(\theta) d\theta$$

$$= \int q(\mu) q(\tau) \Big[-\frac{\lambda_0 \tau}{2} (\mu - \mu_0)^2 - b_0 \tau + \frac{1}{2} \log \frac{\tau}{2\pi} + \frac{1}{2} \log \lambda_0$$

$$+ a_0 \log b_0 + (a_0 - 1) \log \tau - \log \Gamma(a_0) \Big] d\mu d\tau$$

$$= \int q(\mu) q(\tau) \Big[-\frac{\lambda_0 \tau}{2} (\mu - \mu_0)^2 - b_0 \tau + \left(a_0 - 1 + \frac{1}{2}\right) \log \tau - \frac{1}{2} \log 2\pi$$

$$+ a_0 \log b_0 + \frac{1}{2} \log \lambda_0 - \log \Gamma(a_0) \Big] d\mu d\tau$$

$$= E_\mu E_\tau \left[-\frac{\lambda_0 \tau}{2} (\mu - \mu_0)^2 \right] + E_\tau \left[- b_0 \tau + \left(a_0 - 1 + \frac{1}{2}\right) \log \tau \right]$$

$$- \frac{1}{2} \log 2\pi + a_0 \log b_0 + \frac{1}{2} \log \lambda_0 - \log \Gamma(a_0)$$

$$\therefore \mathcal{F} = E_q[\log p(y|\theta)] - KL(q \| p)$$

$$= E_q[\log p(y|\theta)] + \cancel{E_q}\, E_q[\log q] - E_q[\log p(\theta)]$$

$$= \frac{N}{2} E_\tau[\log \tau] - \frac{N}{2} \log 2\pi - E_\tau[\tau] E_\mu \left[\frac{1}{2} \sum_{n=1}^{N} (y_n - \mu)^2 \right]$$

$$- E_\tau[\log q(\tau)] - E_\mu[\log q(\mu)] + E_\mu \left[-\frac{\lambda_0}{2} E_\tau[\tau] (\mu - \mu_0)^2 \right]$$

$$+ E_\tau[-b_0 \tau] + E_\tau[\log \tau] \left(a_0 - 1 + \frac{1}{2} \right) - \frac{1}{2} \log 2\pi + a_0 \log b_0$$

$$+ \frac{1}{2} \log \lambda_0 - \log \Gamma(a_0)$$

$$\mathcal{F} = \left(\frac{N+1}{2} + a_0 - 1\right) E_\tau[\log \tau] - E_\tau[\tau]\left(2b_0 + E_\mu\left[\sum_{n=1}^{N}(y_n-\mu)^2\right]\right)$$

$$- \left(\frac{N+1}{2}\right)\log 2\pi - \log\Gamma(a_0) + a_0 \log b_0 + \frac{1}{2}\log\lambda_0$$

$$- E_\mu[\log q(\mu)] - E_\tau[\log q(\tau)]$$

Ex 5.2

a)

```
%Exercise (a)

mu = 1;
tau = 1;
N = 10;
epsilon = randn(N, 1);
y = mu + epsilon;
disp(y);
```

b) c) d) and e) with parameters initialization to the prior.

```
%Exercise (b) (c) (d) (e) (prior initial values)

% Prior hyperparameters
mu0 = 0;
lambda0 = 3;
a0 = 2;
b0 = 2;
N = length(y);
ybar = mean(y);

% Initial values
m = mu0;
s2 = b0/(a0*lambda0);
a = a0;
b = b0;

% Define log-gamma function for numerical stability
lgamma = @(x) gammaln(x);

% Iterate until convergence
maxiter = 1000;
threshold = 10e-3;
free_energy = zeros(maxiter, 1);
for iter = 1:maxiter
    s2 = 1 / ((a/b) * (N + lambda0));
    m = ((lambda0 * mu0) +  N*ybar) / (lambda0 + N);
    mu = normrnd(m, sqrt(s2));
    a = a0 + (N + 1) / 2;
    b = b0 + (1/2) * (sum((y - m).^2) + lambda0 * (mu0 - m)^2 + (N + lambda0) * s2);
    tau = gamrnd(a, 1 / b);
    s = sqrt(s2);
    free_energy(iter) = -a*log(b) + lgamma(a) - lgamma(a0) + a0*log(b0) + 0.5*log(lambda0) + log(s) - (N/2)*log(2*pi) + 0.5;
    if iter > 1 && abs(free_energy(iter) - free_energy(iter-1)) < threshold
        break;
    end
end

fprintf('Final estimate of m: %f\n', m);
fprintf('Final estimate of s2: %f\n', s2);
fprintf('Final estimate of mu: %f\n', mu);
fprintf('Final estimate of a: %f\n', a);
fprintf('Final estimate of b: %f\n', b);
fprintf('Final estimate of tau: %f\n', tau);
fprintf('Final estimate of tau_mean: %f\n', (a/b));
fprintf('Free energy: %f\n', free_energy(iter));
```

```
Final estimate of m: 0.309630
Final estimate of s2: 0.037901
Final estimate of mu: 0.442003
Final estimate of a: 7.500000
Final estimate of b: 3.679477
Final estimate of tau: 2.063539
Final estimate of tau_mean: 2.038333
Free energy: -10.626591
```

Parameters initialization at random and results.

```
% Initial values
n = 4;
x = 1;
y = 10;
z = x + (y-x)*rand(n,1);
m = z(1);
s2 = z(2);
a = z(3);
b = z(4);
```

```
Final estimate of m: 2.500000
Final estimate of s2: 3.845781
Final estimate of mu: 0.700218
Final estimate of a: 3.000000
Final estimate of b: 47.191563
Final estimate of tau: 0.031926
Final estimate of tau_mean: 0.063571
Free energy: -8.679348
```

If I could only report the results from one run, I would choose the one with the largest free energy as it provides a more accurate explanation of the data.